On Principle of Fermat in refraction of light

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Abstract

The Principle of Fermat represents a unification of the laws of geometrical optics, viz, the laws of rectilinear propagation, reflection and refraction. It states that the optical path taken by light (and hence the time of transit) in propagating between two points, either directly, or via reflection or refraction, is a minimum. Apparent violations of Principle of Fermat have been widely publicized in reflection but less conspicuously so in refraction. This paper examines the latter problem of validity of Principle of Fermat when light passes from a point in one medium to another point in a second medium via refraction. The surface of separation of the two media is found to be an oval, whose parametric equation is obtained. The part of the oval on which actual refraction could take place is determined. The law of Snell of refraction is obeyed at every point on the realistic part of the oval and the optical path is stationary. The curvature of the oval is calculated. If the oval at any point is replaced by a convex surface having a curvature greater than that of the oval, then the optical path is shown to be maximum. Principle of Fermat is modified in such a case to include maximum optical paths. However, the need for such a modification is unnecessary. It is argued that the law of reflection (or refraction) takes place at surfaces where they are locally flat and it is there where Principle of Fermat of minimum optical path is also valid.

Keywords: Principle of Fermat, reflection, refraction, oval, curvature.

Resumen

El Principio de Fermat representa una unificación de las leyes de la óptica geométrica, es decir de las leyes de la propagación rectilínea, la reflexión y la refracción. Se afirma que la trayectoria óptica tomada por la luz (y por tanto el tiempo de tránsito) al propagarse entre dos puntos (ya sea directamente o por medio de la reflexión o refracción) es mínima. Las aparentes violaciones al principio de Fermat se han difundido ampliamente respecto a la reflexión, pero menos notoriamente en lo que concierne a la refracción. Este artículo examina el último problema de la validez del principio de Fermat, cuando luz refractada pasa desde un punto en cierto medio, hacia otro punto situado en un segundo medio. Se encontró que la superficie de separación entre los dos medios es un óvalo, del cual se obtuvo su ecuación paramétrica; y se determinó la parte del óvalo en la cual la refracción podría tener lugar. La ley de refracción de Snell se cumple en cada punto de dicha parte del óvalo, y la trayectoria óptica resultante es estacionaria, además se calculó la curvatura del óvalo. Si cualquier punto del óvalo se sustituye por una superficie convexa que tenga una curvatura mayor que la del óvalo, entonces la trayectoria óptica se maximiza. En tal caso, el principio de Fermat se modifica para incluir trayectorias ópticas máximas. Sin embargo, la necesidad de tal modificación es innecesaria. Se argumenta que la ley de la reflexión (o de refracción) tiene lugar en superficies localmente planas, y es allí donde la trayectoria óptica mínima del principio de Fermat es válida.

Palabras clave: Principio de Fermat, reflejo, refracción, oval, curvatura.

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I. INTRODUCTION

The laws of rectilinear propagation of light, reflection and refraction are fundamental laws upon which geometrical optics is based. In 1658, Pierre de Fermat enunciated his principle which states that the optical path (or equivalently, the time taken) by light in propagating between two points, either directly, or via reflection or refraction, is a minimum.

It was shown that the laws of rectilinear propagation, reflection and refraction can all be derived from Principle of Fermat. Special cases of apparent violation of Principle of Fermat have been widely publicized in the literature [1, 2, 3, 4, 5, 6]. They are all concerned with the solitary example of reflection of light inside an elliptical reflector when light passed from one focus to the other via reflection [1, 2, 3, 4, 5, 6]. By the special properties of an ellipse, the optical path or time taken by light is a constant, *i.e.*, stationary. If the elliptical reflector is replaced by a concave mirror at the point of reflection with a curvature greater than that of the elliptical surface at that point, then the optical path or time taken is actually shown to be a maximum. A similar situation involving refraction has also been reported in the literature [5, 6] but without much detail. In this paper, we critically examine this latter case in detail, obtain parametric equations of the refracting surface and discuss the applicability of Principle of Fermat in this case.

II. THE PROBLEM

Consider the following problem:

Light travels between a point A in one medium, to another point B in a second medium via refraction, at the interface between the two media. Find the nature of the surface of separation such that the optical path of light (and hence, the time of transit) is a constant, *i.e.*, stationary. It has been reported that the surface is an *oval* [5], which is an egg-shaped curve without a specific definition [7]. The oval is convex towards the rarer medium of the two [5]. Light travelling between the two points A and B via refraction, at a point C on this oval surface will obey the law of Snell law of refraction, and also have the same optical path.

Consequently, a point source located at A will form a sharp image at B devoid of any form of aberration. Such a surface is then called an *aplanatic surface* [5, 6]. If the oval is replaced at the point C by a curved surface having a curvature greater than that of the oval at that point, the optical path (and hence the time taken by light) will actually be a maximum, in apparent violation of Principle of Fermat.

III. THE SOLUTION

It is practicable to assume that the point A lies in air (refractive index ≈ 1) and the point B lies in say, glass (refractive index $n \approx 1.52$). Place A and B on the x-axis of a Cartesian coordinate system with A at a distance α to the left of the origin O and B at a distance β to the right of O (Figure 1). The oval surface to be determined passes through O, at which point it will necessarily be vertical so that the direct ray from A to B passes undeviated at O. The optical path of this direct ray is required to be a constant:

$$\alpha + n\beta = d \,. \tag{1}$$

Let C be a general point on the surface to be determined (Figure 1). Then the optical path for the general ray is:

$$b + na = d = \alpha + n\beta \,. \tag{2}$$

where CA = b and CB = a. With AB = $\alpha + \beta = c$, we apply the law of cosine to Δ ABC getting (Figure 1):

$$c^2 = a^2 + b^2 - 2ab\cos C \,, \tag{3}$$

whence:

$$C = \cos^{-1}\left[\frac{a^2 + b^2 - (\alpha+\beta)^2}{2ab}\right]. \tag{4}$$

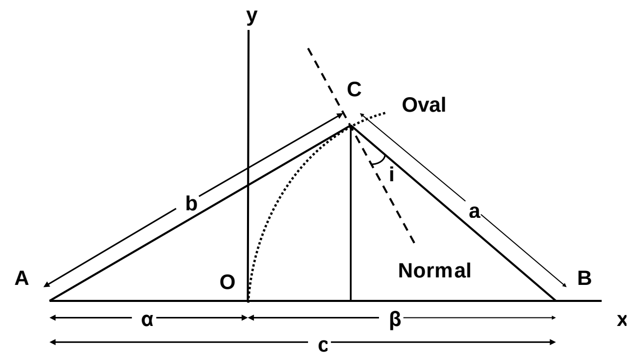

FIGURE 1. Geometry of passage of light from point A in one medium (air) to another point B in a second medium (glass) via refraction at C.

Next, by applying the law of sines to Δ ABC, one gets:

$$\frac{c}{\sin C} = \frac{b}{\sin B} \,. \tag{5}$$

Whence:

$$B = \sin^{-1}\left(\frac{b\sin C}{\alpha + \beta}\right). \tag{6}$$

The coordinates of the point C follow (vide Figure 1):

$$x = \beta - a\cos B \,, \tag{7}$$

and

$$y = a\sin B \,. \tag{8}$$

It should be noted that Equation (8) gives the positive values of the ordinate and therefore represents the upper half of the oval only. For the lower half of the oval, we use instead:

$$y = -a\sin B \,. \tag{9}$$

Since all quantities in Equations (7, 8, 9) are either constants or functions of b, they represent parametric equations of the oval sought for.

The equations (7, 8, 9) are exact and the values of the coordinates can be conveniently calculated using any software like Excel, or even by a hand-held calculator. As an example, we have obtained the oval with the values of α = 10 cm and β = 10 cm, which is shown in Figure 2. The oval indeed resembles the shape of an egg with its pointed side towards A. However, the only realistic front part of the oval is shown by the solid line in Figure 2. The remainder of the oval given by the dotted line lies in the 'shadow region' where light from A cannot possibly reach.

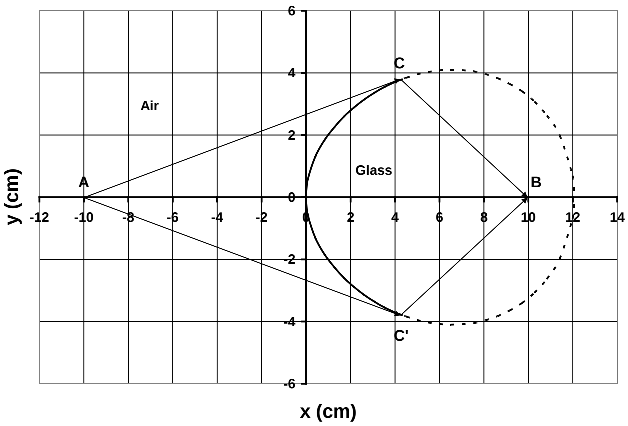

FIGURE 2. Surface of stationary optical path in problem of Figure 1. Rays between AC and AC′ will form a sharp image of A at B without any aberration. Law of Snell will be obeyed at every point on the surface COC′.

The upper limit of the realistic part of the oval is found when the ray AC is tangential to the oval at C. This happens when the angle A in Fig. 1 is a maximum and

$$A = A_m = \tan^{-1}\left(\frac{y_m}{\alpha + x_m}\right). \tag{10}$$

Alternatively, by virtue of *the Principie of reversibility of light*, the same condition is obtained when the angle of incidence i in Figure 1 is the critical angle i_c = C – 90º for refraction of light from glass to air:

$$i = i_c = \sin^{-1}\left(\frac{1}{n}\right) = 41.14^{o}\cdot \tag{11}$$

Both methods give the coordinates of the limiting point C as (4.354, 3.806). By symmetry, the coordinates of the point C′ in the lower branch is (4.354-3.806). Only rays between AC and AC′ are able to reach the point B and form a sharp image devoid of any form of aberration. It should be mentioned that law of Snell of refraction will be valid at every point on the surface COC′.

IV. CURVATURE OF THE OVAL

It is instructive to calculate the curvature at any point on the oval. In cartesian coordinates, the radius of curvature R is given by [8]:

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}. \tag{12}$$

The curvature κ is the reciprocal of R:

$$\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}. \tag{13}$$

The curvature is a measure of the second derivative at a point on the curve. For the oval, the radius of curvature and the curvature at a point may be determined numerically by calculating the first and second derivatives using the central difference scheme and substituting them in the Equations (12) and (13). Alternatively, the equation of the circle passing through three consecutive points can be used:

$$\sqrt{(x-a)^2 + (y-b)^2} = R\,, \tag{14}$$

where (a, b) are the coordinates of the center of the circle.

The three unknowns a, b and R can be found from the three equations at the three consecutive points (x_1, y_1), (x_2, y_2) and (x_3, y_3) to yield the curvature at the middle point.

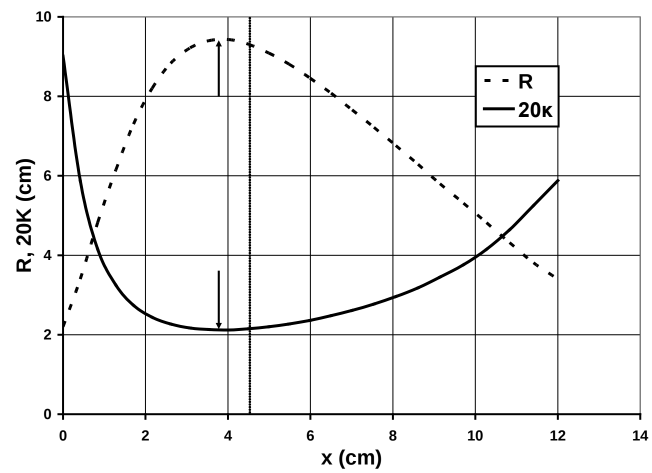

FIGURE 3. Radius of curvature R and curvature κ of oval of Figure 2. The arrows mark their extremum locations. The practical part of the oval lies to the left of the dotted line.

The radius of curvature R and the curvature κ thus obtained are plotted in Figure 3. R attains its maximum values at a distance of x = 3.827 cm from O and κ attains its minimum values at the same points. The dotted line in Figure 3 marks the rightward limits of the oval where refraction can actually take place. Following the arguments of the case of reflection at the elliptical mirror [1, 2, 3, 4, 5, 6], if the oval is replaced by a convex refracting surface having a curvature greater than that of the oval at any point, then the optical path for passage of light between A and B will actually be a maximum in apparent violation of Principle of Fermat.

V. DISCUSSION

It is well known that the original statement of Principle of Fermat holds true for reflection and refraction of light at plane surfaces [4]. It is only with curved surfaces that there seem to be apparent violations of the original statement [1, 2, 3, 4, 5, 6]. The need for modifications of Principle of Fermat have thus been made for reflection [1, 2, 3, 4, 5, 6] and for refraction [5, 6]. Suggested modifications call for the inclusion of maximum optical path [1, 2, 3, 4, 5, 6] or stationary optical path [1, 4, 5] in the statement. We now take a closer scrutiny at these modifications. We note that the first derivative merely gives the slope of a curve at a point whilst it is the second derivative which gives a measure of the curvature. Thus to the first approximation, a curve (or surface) is locally flat. And if the law of reflection (or refraction) holds at a point, then Principle of Fermat in its original form also does, at least locally. If the optical path is stationary in reflection (as in the case of the elliptical reflector) or refraction (as in the case of the oval refracting surface), then Principle of Fermat may be modified to include stationary optical paths (even if that pertains to curved surfaces.) However, the inclusion of maximum optical paths in order to mitigate the violation of Principle of Fermat is questionable. Since the locations of the purported violations are away from where the reflection (or refraction) takes place and it can be seen that at such places, the law of reflection (or refraction) will not be obeyed and light will not traverse along such paths.

Stated otherwise, it is unfair to find violations of Principle of Fermat where reflection (or refraction) does not take place. Hence the inclusion of maximum optical path in Principle of Fermat is unnecessary. In closing, it may be stated that Principle of Fermat represents a unification scheme for the laws of geometrical optics, Figure 3, the laws of rectilinear propagation, reflection and refraction. It also layed the foundation upon which variational calculus was to be developed a century later.

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