A further look at capacitors in complex arrangements

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Abstract

Recently, a previous contributor showed how the delta-wye conversion may be applied to find the equivalent capacitance of five capacitors arranged in a bridge configuration. In fact, that contributor gave an algorithm for this purpose which showed that the equivalent capacitance depends upon all five capacitors. However, in this paper, we point out that there is a special case where the equivalent capacitance does not depend upon one of the five capacitors.

Keywords: Combination of capacitors, Bridge circuits, Delta-wye and wye-delta conversions.

Resumen

Un contribuidor anterior recientemente demostró cómo usar la conversión Delta-Estrella para encontrar la capacitancia equivalente de cinco capacitores en una configuración de puente. De hecho, ese contribuidor presentó un algoritmo para este propósito y mostró que la capacitancia equivalente depende de los cinco capacitores. Sin embargo, en este artículo mostramos que existe un caso especial para el cual la capacitancia equivalente no depende de uno de los cinco capacitores.

Palabras Clave: Combinaciones de capacitores, Circuitos puente, Conversiones delta-estrella y estrella-delta.

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I. INTRODUCTION

In [1], Rizhov investigated the problem of finding the equivalent capacitance C_0 (with respect to terminals T_3 and T_4) of five capacitors connected in a bridge configuration as shown in Figure 1.

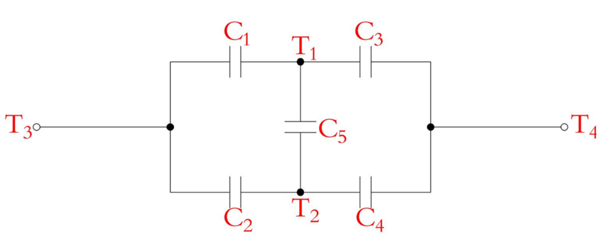

FIGURE 1. Five capacitors connected in a bridge configuration.

His approach was to arrive at a system of seven unknowns, which were solved for one particular case. However, as pointed out by Atkin [2], this method "did not reveal how a general solution could be obtained, giving C_0 as a function of the five capacitances". On the other hand, Atkin [2] showed how the delta-wye conversion may be applied for this very purpose. (Incidentally, the use of delta-wye and wye-delta conversions to solve problems of this sort is a well-known technique described in many electrical circuit analysis textbooks [3, 4]).

Unfortunately, the algorithm given by Atkin in [2] hides the fact that C_0 is independent of the value of C_5 if:

$$\frac{C_3}{C_1}=\frac{C_4}{C_2}=C.$$

In fact, there is no hint at all in [1] or [2] that the equivalent capacitance might not depend upon C_5 It is the purpose of this paper to demonstrate this fact in two ways: by using i the algorithm given by Atkin [2] and ii the concept of a balanced bridge circuit.

II. REVIEW OF THE METHOD DESCRIBED BY ATKIN

In Atkin's method [2], the delta configuration of C_1, C_2 and C_5 in Figure 1 is first converted to a wye circuit, given by C_6, $C_7$and C_8 of Figure 2i, where:

$$C_6=C_1+C_5+\frac{C_1C_5}{C_2}, \qquad (1a)$$

$$C_7=C_2+C_5+\frac{C_2C_5}{C_1}, \qquad (1b)$$

and:

$$C_8=C_1+C_2+\frac{C_1C_2}{C_5}. \qquad (1c)$$

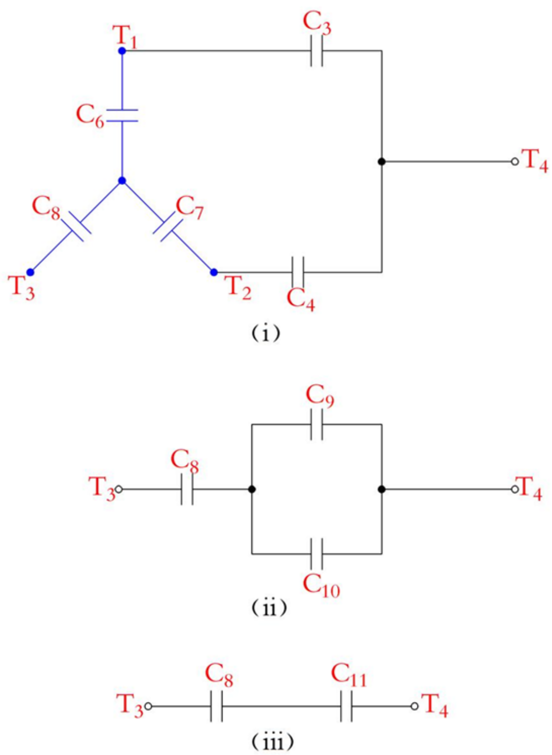

FIGURE 2. Simplification of Figure 1 using a delta-wye transformation.

Comparing Figure 2i with Figure 2ii shows that C_9 is the series combination of C_3 and C_6 and C_{10} is the series combination of C_7 and C_4 Hence:

$$C_9=\frac{C_3C_6}{C_3+C_6}, \qquad (2a)$$

and:

$$C_{10}=\frac{C_4C_7}{C_4+C_7}. \qquad (2b)$$

Furthermore, comparing Figure 2ii with Figure 2iii shows that C_{11} is the parallel combination of C_9 and C_{10} Hence:

$$C_{11}=C_9+C_{10}. \qquad (3)$$

Finally, from Figure 2iii, it is clear that C_0 is the series combination of C_8 and C_{11}

Hence:

$$C_0=\frac{C_8C_{11}}{C_8+C_{11}}. \qquad (4)$$

III. DETERMINING THE EQUIVALENT CAPACITANCE FOR THE SPECIAL CASE WHERE $\frac{C_3}{C_1}=\frac{C_4}{C_2}$

A. Using the algorithm given by Atkin

From Equations 2a and 1a:

$$C_9=\frac{C_3}{1+\frac{C_3}{C_6}}=\frac{C_3}{1+\frac{C_3}{C_1+C_5+\frac{C_1C_5}{C_2}}},$$

$$=\frac{C_3}{1+\frac{C_3/C_1}{1+\frac{C_5}{C_1}+\frac{C_5}{C_2}}},$$

$$=\frac{C_3}{1+\frac{C}{K}},$$

$$=\frac{KC_3}{K+C}, \qquad (5)$$

where $K=\left(\frac{C_1+C_2}{C_1C_2}\right)C_5+1$, and $C=\frac{C_3}{C_1}$.

Similarly, from Equations 2b and 1b:

$$C_{10}=\frac{C_4}{1+\frac{C_4}{C_7}}=\frac{C_4}{1+\frac{C_4}{C_2+C_5+\frac{C_2C_5}{C_1}}},$$

$$=\frac{C_4}{1+\frac{C_4/C_2}{1+\frac{C_5}{C_2}+\frac{C_5}{C_1}}}, \qquad (6)$$

$$=\frac{C_4}{1+\frac{C}{K}},$$

$$=\frac{KC_4}{K+C},$$

With:

$$C=\frac{C_4}{C_2}.$$

Substituting Equations 5 and 6 into Equation 3 gives:

$$C_{11}=\frac{K}{K+C}\left(C_3+C_4\right). \tag{7}$$

Furthermore, Equation (1c) can be written as:

$$\begin{aligned} C_8&=\left(\left(\frac{C_1+C_2}{C_1C_2}\right)C_5+1\right)\frac{C_1C_2}{C_5},\\ &=K\frac{C_1C_2}{C_5}. \end{aligned} \tag{8}$$

Substituting Equation (7) and Equation (8) into Equation (4) gives:

$$\begin{aligned} C_0&=\frac{K^2\frac{C_1C_2}{C_5}\frac{1}{K+C}\left(C_3+C_4\right)}{K\frac{C_1C_2}{C_5}+\frac{K}{K+C}\left(C_3+C_4\right)},\\ &=\frac{K\frac{C_1C_2}{C_5}\frac{1}{K+C}\left(C_3+C_4\right)}{\frac{C_1C_2}{C_5}+\frac{1}{K+C}\left(C_3+C_4\right)}. \end{aligned} \tag{9}$$

Further simplification of Equation (9) gives:

$$\begin{aligned} C_0&=\frac{\frac{C_1C_2}{C_5}K\left(C_3+C_4\right)}{\frac{C_1C_2}{C_5}\left(K+C\right)+\left(C_3+C_4\right)},\\ &=\frac{K\left(C_3+C_4\right)}{K+C+\left(C_3+C_4\right)\frac{C_5}{C_1C_2}},\\ &=\frac{K\left(C_3+C_4\right)}{1+C+\left(C_1+C_2+C_3+C_4\right)\frac{C_5}{C_1C_2}},\\ &=\frac{K\left(C_3+C_4\right)}{\left(1+C\right)\left[1+\left(\frac{C_1+C_2+C_3+C_4}{1+C}\right)\frac{C_5}{C_1C_2}\right]},\\ &=\frac{K\left(C_3+C_4\right)}{\left(1+C\right)\left[1+\left(\frac{C_1\left(1+C\right)+C_2\left(1+C\right)}{1+C}\right)\frac{C_5}{C_1C_2}\right]}. \end{aligned} \tag{10}$$

Hence:

$$\begin{aligned} C_0&=\frac{K\left(C_3+C_4\right)}{\left(1+C\right)K},\\ &=\frac{C_3}{1+C}+\frac{C_4}{1+C},\\ &=\frac{C_3}{1+\frac{C_3}{C_1}}+\frac{C_4}{1+\frac{C_4}{C_2}},\\ &=\frac{C_3C_1}{C_1+C_3}+\frac{C_4C_2}{C_2+C_4}. \end{aligned} \tag{11}$$

As can be seen from Equation (11), the equivalent capacitance does not depend upon C_5 as claimed. In fact, the value given by Equation (11) is the same value that would be given if C_5 is not connected to the circuit. Hence, if $\frac{C_3}{C_1}=\frac{C_4}{C_2}=C$, the bridge circuit behaves the same regardless of the value of C_5.

B. Using the concept of the balanced bridge

As can be seen from the above, it is straightforward but tedious to derive the equivalent capacitance for this special case. Fortunately, there is a well-known alternative way of deriving this same result using the idea of a balanced bridge [3, 4]. For completeness, we will review this method very briefly.

Consider the bridge circuit of Figure 3. If the current through Z_5 is zero, the bridge is said to be balanced and Z_5 will have no effect on the bridge circuit. In fact, Z_5 can now be an open or short circuit.

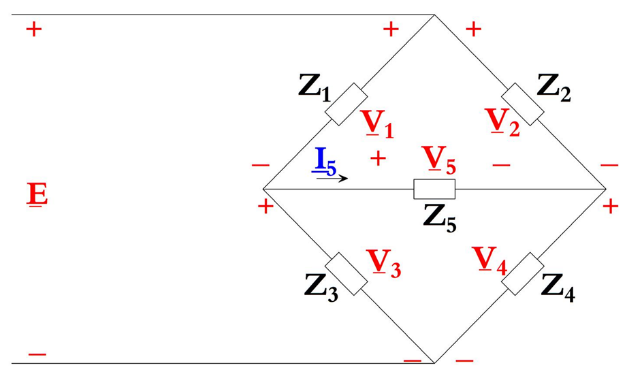

FIGURE 3. General bridge circuit with arbitrary impedances. Note the underscore indicates a phasor voltage or current.

If we replace Z_5 with an open circuit, *i.e.,* remove Z_5 the total impedance will be given by the parallel combination of Z_1+Z_3 and Z_2+Z_4 giving rise to Equation 11.

Furthermore, to ensure that the bridge is balanced, it is easily shown that $\frac{Z_1}{Z_3}=\frac{Z_2}{Z_4}$, or equivalently, $\frac{C_3}{C_1}=\frac{C_4}{C_2}$.

To see this, note that the current through Z_5 is zero when the voltage across Z_5 is zero, *i.e.:*

$$\underline{V}_5=\underline{V}_3-\underline{V}_4=0. \qquad (12)$$

Furthermore, when $\underline{I}_5=0$, we apply the voltage divider rule to obtain:

$$\underline{V}_3=\frac{Z_3}{Z_1+Z_3}\underline{E}=\frac{1}{1+Z_1/Z_3}\underline{E}, \qquad (13)$$

and:

$$\underline{V}_4=\frac{Z_4}{Z_2+Z_4}\underline{E}=\frac{1}{1+Z_4/Z_2}\underline{E}. \qquad (14)$$

Substituting Equation 13 and 14 into Equation 12 gives the desired result:

$$\frac{Z_1}{Z_3}=\frac{Z_2}{Z_4}.$$

IV. CONCLUSIONS

We have shown that the equivalent capacitance of five capacitors connected in a bridge configuration does not depend upon one of the capacitors for a special case. We showed this by using the algorithm provided by Atkin and also by using the concept of a balanced bridge.

REFERENCES

[1] Rizhov, A., *Computation of capacitors in complex arrangements*, Physics Education **46**, 551-553 (2011).

[2] Atkin, K., *An alternative approach to capacitors in complex arrangements*, Physics Education **47**, 326-328 (2012).

[3] Robbins, A. H. & Miller, W. C., *Circuit analysis: theory and practice*, 3rd Ed. (Thomson Delmar Learning, New York, 2004).

[4] Boylestad, R. L., *Introductory Circuit Analysis*, 9th Ed. (Prentice Hall, New Jersey, USA, 2000).