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Pendulum motion on modified trajectories The cycloidal pendulum as an old-time GPS

Imre Beszeda, Tamás Stonawski
University of Nyíregyháza, Nyíregyháza, 4400 Sóstói Street 31, Hungary.

E-mail: beszedaimre@gmail.com, stonawski@gmail.com

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Abstract
According to physics textbooks, measurements show that the period of a simple gravity pendulum does not change significantly at small displacement angles (up to about 5°), but increases at larger displacements. For small swings, i.e. for angles less than 5°, the constant period is explained by the fact that in this case the curvature of the pendulum's trajectory is negligible, the pendulum approximates a simple harmonic oscillator (where the oscillation time is independent of the amplitude). This 5° limit is not really based on the results obtained from measurements, but rather on a differential equation (which could be difficult to explain in primary and secondary school in the absence of mathematical background), where the small angle approximation sin $\alpha \approx \alpha$ can be applied below 5°, and so we get a much more simple equation. We wanted to check where this limit might be in the measurements and whether this amplitude dependence could actually be measured below this value.

Keywords: Classical Mechanics teaching.

Resumen
Según los libros de texto de física, las mediciones muestran que el período de un péndulo de gravedad simple no cambia significativamente en ángulos de desplazamiento pequeños (hasta unos 5°), pero aumenta en desplazamientos más grandes. Para oscilaciones pequeñas, es decir, para ángulos inferiores a 5°, el período constante se explica por el hecho de que en este caso la curvatura de la trayectoria del péndulo es despreciable, el péndulo se aproxima a un oscilador armónico simple (donde el tiempo de oscilación es independiente de la amplitud). Este límite de 5° no se basa realmente en los resultados obtenidos de las mediciones, sino más bien en una ecuación diferencial (que podría ser difícil de explicar en la escuela primaria y secundaria en ausencia de antecedentes matemáticos), donde la aproximación del ángulo pequeño sen $\alpha \approx \alpha$ se puede aplicar por debajo de 5°, y así obtenemos una ecuación mucho más simple. Queríamos verificar dónde podría estar este límite en las mediciones y si esta dependencia de amplitud realmente podría medirse por debajo de este valor.

Palabras clave: Enseñanza de la Mecánica Clásica.

I. PRELUDE

According to physics textbooks, measurements show that the period of a simple gravity pendulum does not change significantly at small displacement angles (up to about 5°), but increases at larger displacements. For small swings, i.e. for angles less than 5°, the constant period is explained by the fact that in this case the curvature of the pendulum's trajectory is negligible, the pendulum approximates a simple harmonic oscillator (where the oscillation time is independent of the amplitude).

This 5° limit is not really based on the results obtained from measurements, but rather on a differential equation (3) (which could be difficult to explain in primary and secondary school in the absence of mathematical background), where the small angle approximation sin $\alpha \approx \alpha$ can be applied below 5°, and so we get a much more simple equation. We wanted to check where this limit might be in the measurements and whether this amplitude dependence could actually be measured below this value.

Our measurements were carried out with a microcontroller with a theoretical accuracy of 4 microseconds (but in practice less accuracy, see the description of the measurements later).

For technical reasons, with a pendulum of length about 1 meter we could not measure at amplitudes less than 2° (Figure 1), but above this value the period was not constant. In order to be able to reduce the amlitude below 2°, we had to increase the length of the pendulum. We chose for this purpose a previously made 573cm long Foucault pendulum. Below the amplitude of 2°, a constant period could be observed (Fig. 2). In order to observe this phenomenon for even shorter pendulums (and not to have to work with such an uncomfortably long pendulum) it is worth first focusing on the mathematical description, without considering the above-mentioned approximation.

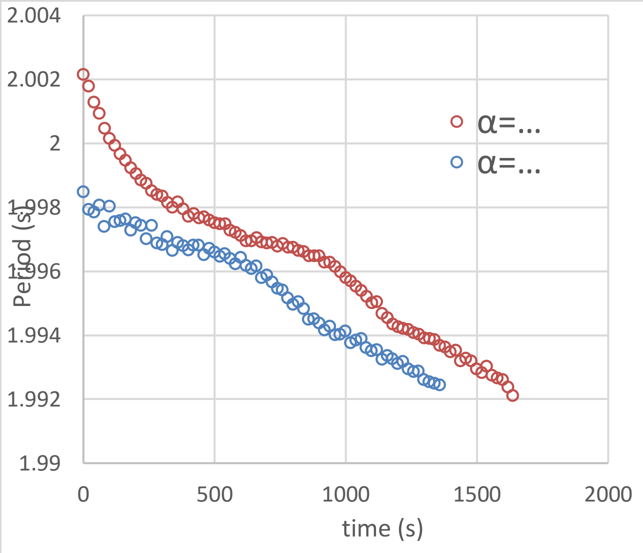

FIGURE 1. Variation of the oscillation time of a 98 cm mathematical pendulum as a function of time.

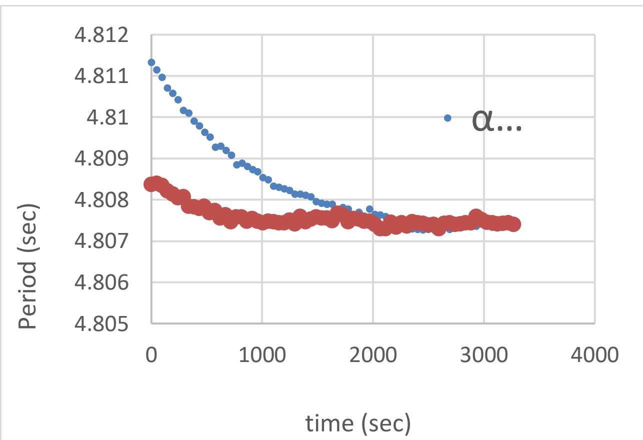

FIGURE 2. Variation of the period of a 573 cm long pendulum as a function of time.

II. AMPLITUDE DEPENDENCE OF THE SIMPLE GRAVITY PENDULUM

In one of his main topics, Galileo Galilei wanted to find timekeeping devices with suitable accuracy to investigate the rules of freefall. In addition to his special water meter, he realized during studying the pendulum motion that the period of a simple gravity pendulum is independent of the bob's weight and material, and for small amplitudes, of the amplitude as well.

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad (1)$$

Limitation to small angles is the result of the small angle approximation sinα≈α in the equation of pendulum motion. This can be seen writing the tangential acceleration of the pendulum as:

$$ma_t = -mg\, sin\alpha, \qquad (2)$$

$$ml\frac{d^2\alpha}{dt^2} = -mg\, sin\alpha. \qquad (3)$$

The solution of the differential equation (3) can be given muc ould like to find a general solution, with the solution of an elliptic integral we get the Wallis formula [1]:

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 sin^2\frac{\alpha_0}{2} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 sin^4\frac{\alpha_0}{2} + \cdots\right] = 2\pi\sqrt{\frac{l}{g}}\cdot\sum_{n=0}^{\infty}\left(\left(\frac{(2n)!}{(2^n\cdot n!)^2}\right)^2 sin^{2n}\frac{\alpha_0}{2}\right). \qquad (4)$$

Equation (4) shows the amplitude dependence of the period. The relative error is 0.05% at the initial amplitude α_0 = 5°, 1% at α_0 = 22° and 18% at α_0 = 90° (Fig. 3). Since the motion of a real pendulum is damping motion, the periode is decreasing. If we do not give the pendulum a certain external push at the right time, which would ensure a constant amplitude (this is not investigated here), the pendulum would have a decreasing amplitude (Fig. 3). Arriving at the ideal small amplitude, we could use it to measure only for short time intervals due to further attenuations.

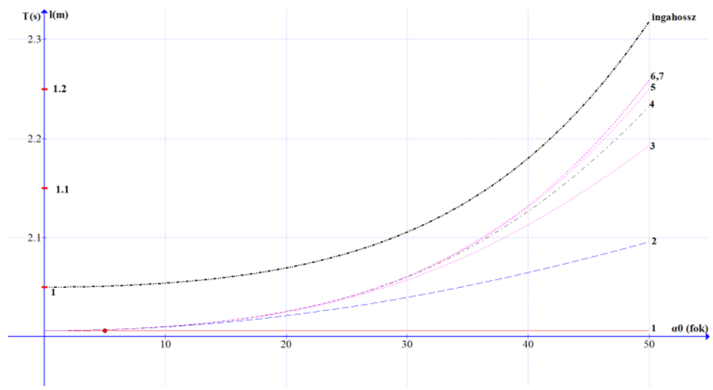

FIGURE 3. The amplitude dependence of the gravity pendulum according to Equation (4) by adding the terms (1, 2, 3 and 4) in parentheses (l = 1m, g = 9.81m/s^2). Considering 6 terms at 50° approximates well the general expression (4). The graph shows clearly why the expression (1) can be used for amplitudes less than 5°. The graph also shows how the pendulum length should change with time to have an amplitude-independent periode.

Looking at the pendulum length graph in Fig. 3, we can conclude that if we could reduce somehow the pendulum length, according to the values of the graph (the higher amplitude the more intense decrease in the length), we could produce an amplitude-independent pendulum. A simple solution could be to use obstacles (needles or screws) at different swing angles, e.g. at every 10 degrees in the path of the cord: this would shorten its length.

amplitude (degree)	0	10	20	30	40	50
degree of shortening of the lengt (mm)	0	4,1	19,4	55,6	130,1	268,3

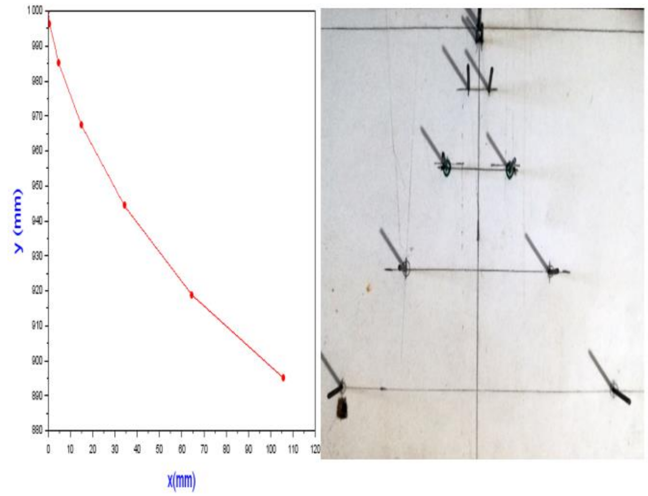

FIGURE 4. Left: Coordinates of the obstacles, calculated according to the table above, from 0 to 50 degrees (up to 20 degrees is sufficient in practice). Right: A photo of the pivot and obstacles (needles) positioned according to the points on the graph.

We prepared a 99 cm long pendulum with a pivot and needle-obstacles as shown in Fig. 4. Its period vs. time plot is shown in Figure 5, which shows clearly that below 5° its period can be considered constant with a good approximation (decreased from 1.993s to 1.991s).

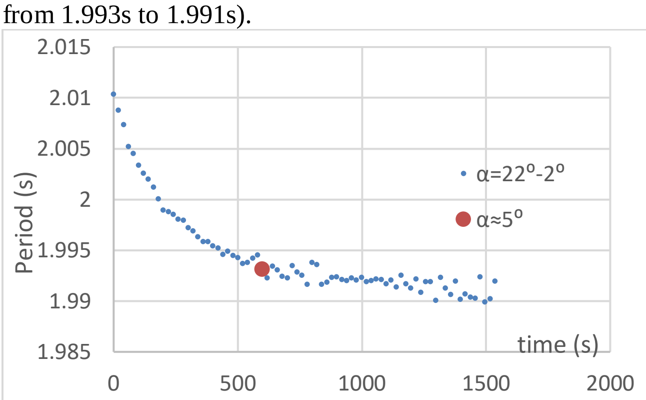

FIGURE 5. Period vs. time plot of the pendulum (with needle-obstacles) of 99 cm length.

Thus, obstacles shorten the length of the pendulum. In practice, however, this implementation is not accurate enough, as the shortening is slightly different due to the roughness of the resulting broken line. (A more accurate solution could be to reduce the length of the broken lines / increase the number of needles). More accurate calculations (i.e., finding the equation of the ideal curve), however, partially or completely exceed the level of mathematical knowledge in high school.

Below, however, we would like to find a curve on which the pendulum bob could move with an amplitude independent period, i.e., no matter what amplitude we started, the period would not change. This would mean that two pendulums started with different amplitudes, would reach the vertical position in the same time. This property is called tautochronous. (The tautochron property can also be found in solving the brachistochron problem.) Huygens proved in his theorem XXV in 1659 that such an tautochron trajectory is cyclois [2].

The parametric equation of the cyclois is:

$$x = r(t - \sin(t)),$$

$$y = r(1 + \cos(t)). \qquad (5)$$

If, on the other hand, the pendulum moves on a cycloic (not circular) trajectory, then the constant relation l = r = is no longer valid (r = constant: the radius of the circle that generates the cycloist, $l \neq$ constant: the pendulum length, which is amplitude-dependent). The cyclois trajectory can be achieved by applying plates with certain curvature so that the shape of the cyclois will determine the trajectory of the pendulum, as well. Another merit of Huygens on this topic is that he was able to prove that, using a cyclois-shaped plate, the trajectory of the pendulum bob will also be cycloic (a curve that played an important role in the seventeenth-century mathematics) [3]. This fact follows from the mathematical theorem that the evolution of a cyclois (the set of centers of its curvature) is itself cycloic. In this case, the first part of the pendulum cord lies on the plate and the rest is tensioned tangentially (Fig. 6). Since we can give different values of r based on (5), different cycloids with different curvatures can be obtained. The question arises, which cyclois will be good for our pendulum to be made? There is the following relationship between the period (or pendulum length) and the cyclois equation [4]:

$$r = \frac{T^2}{\pi^2} g. \qquad (6)$$

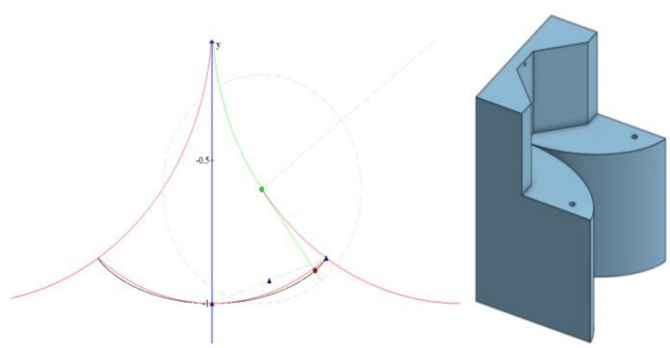

FIGURE 6. Left: Construction of the cyclois trajectory with its evolute. The editing also suggests the technical implementation: two cyclois plates can force the pendulum into this trajectory. Edited with Graph [5]. Right: 3D printed suspension [6], which is suitable for simultaneous examination of three bifilarly suspended pendulums (2 cycloidal and one circular).

It can be seen from this that one cyclois can be edited for a pendulum (or we are looking for the appropriate pendulum length for a given cyclois).

Huygens's initial motivation initiated from noticing that the period of pendulum motion was amplitude-dependent: the periods of the smallest and 90-degree amplitudes were proportional to each other as 29/34. Huygens tried to compensate the variable period first by reducing the length of the pendulum at large amplitudes by positioning certain obstacles (similarly as written above). Empirically determined barrier positions, on the other hand, did not provide much more accurate values. It was many years later he was able to theoretically determine the shape of this suspension.

Huygens thus implemented the plan (from 1656 to 1693) that Galileo Galilei dealt with in the last years of his life: he built a pendulum clock that could even be used as a marine chronometer. Construction of an accurate clock meant the creation of a device by which longitude could be determined on the board of ships in the twentieth century (that can also be considered as an old-time GPS). In a letter dated January 12, 1657, Huygens wrote, "these days I have found a new clock design that can measure the time so accurately that there is a great hope of determining longitude, even if the clock is shipped on the sea".

So if an accurate clock were taken on a ship, the longitude could be determined by comparing the local time of day and the time at home.

It took of course a long time for the technical implementation of such a precise clock (not affected even by the swaying of the ship): in 1714, the tender announced by Queen Anna was won by John Harrison [7], who became a curiosity in the technical history by developing of four different clock constructions with the work of a life.

III. CARRYING OUT THE MEASUREMENT

The cycloidal-shaped suspension profile can also be plotted [8] and then cut out of wood, but can also be 3D-printed [6]. We chose the latter.

To increase the accuracy, the period was measured using the micros() command of an Arduino Nano microcontroller, instead of the manual stopwatch, so we can measure time with a theoretical accuracy of up to 4 microseconds [9] but in practice this accuracy could not be achieved. This may be due to the inaccuracy of the sensor, as it is the cheapest IR sensor available, (but we think it has sufficient accuracy for such a school-measurement). Furthermore, the pendulum stand is not an infinitely rigid body, but moves a little for sure (can wobble, bend, etc.…), so the pendulum does not always enter the field of view of the sensor exactly in the same place and at the same time. Accuracy could have been further increased by reducing the reflective surface area, but the physical dimensions of the 100 gram balance weight were given.

The main part of the measurement is a so-called IR obstacle avoidance sensor placed under the pendulum bob in the equilibrium position, 2cm below its bottom (Fig. 7). The sensor emits IR signals during operation. If there is an object within a few cm of the sensor, it detects the IR signals reflected from it; then the output pin of the sensor gives a zero signal, otherwise 1. This signal was the input at the digital pin D3 of an Arduino Nano microcontroller. (However, be careful with this sensor not to get a lot of sunlight or lamp light, as it may interfere with its operation.)

Based on the above, the principle of our measurement is as follows:

The obstacle (the pendulum) is in front of the sensor, in its detection range (the output signal is 0), and then it swings out of the range (eg to the right), then the output signal changes to 1. Coming back into the range, the output signal will turn to 0, then again going out to the left, the output signal will be 1 again. So we have to read the time every second case when the sensor gives a zero signal.

Instead of the period T we measured 10T to reduce the standard deviation and then T was plotted as a function of time (thus T was obtained theoretically in microseconds). (With a manual stopwatch an accuracy of 0,01 second can be achieved, but this accuracy is greatly decreased by the uncertainty of manual starting and stopping.)

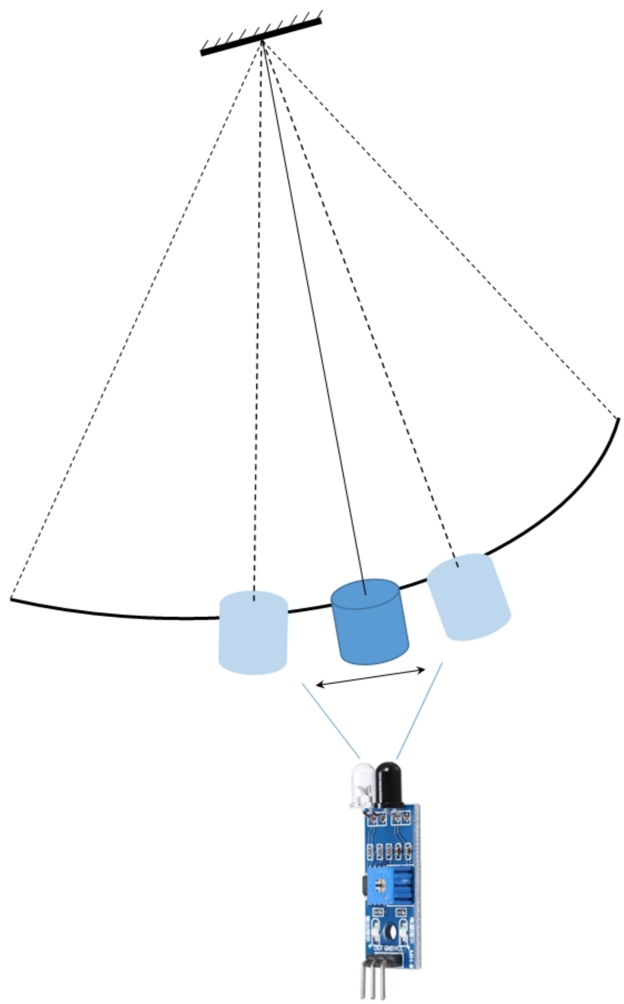

FIGURE 7. Technical diagram of the measurement with the IR obstacle avoidance sensor.

The advantage of our measurement is the much higher accuracy compared to manual measurements (in principle microseconds, but differences in the order of magnitud of 0,0001s only appear as standard deviation), but the disadvantage is that we could not measure amplitudes smaller than 2 cm: this IR sensor has a relatively wide viewing angle, the bob at the end of the pendulum is also quite wide (2.5cm in diameter); these circumstances indicate that the amplitude must be greater than 2 cm in order the pendulums"s bob certainly exit the field of view of the sensor in both directions (Fig. 7).

Thus e.g. in case of a 1 m long pendulum, the 3 cm amplitude means a deviation of 1.72º, so we cannot measure with a smaller amplitude. In the case of the Foucault pendulum (573 cm), on the other hand, a 7 kg iron ball is the weight, at the bottom of which a disk of 4 cm diameter was attached for the measurement, which could be detected by the IR sensor. In this case, the 10 cm amplitude corresponds to a deviation angle of 1º only, i.e. measuring periods of deviation angles much smaller than that of a 1-meter pendulum did not cause a problem either.

A. The code used in the measurements

```
int pin = 3;  // the sensor signal goes to digital input D3
unsigned long elozo_ido;  // the function micros() returns
the number of microseconds since the Arduino board began
running the program
unsigned long ido; // ido will be the last time read, while the
elozo_ido will be the time value read previously
unsigned long delta_t;    // this is the period, which can be
get as the difference of ido and elozo_ido
boolean most_kell_merni = true;   // the value of
most_kell_merni (= the sensor's signal has to be read) is
initially set to true
long i = 0;   // this is a counter that is used to read the time
value at every second pass of the pendulum
void setup()
{
  Serial.begin(9600);   // time values will be written to the
serial monitor (in 10T steps)
  pinMode(pin, INPUT);  // pin D3 is set as input
  elozo_ido = micros(); // at startup, use micros() to read the
current value of time into the variable elozo_ido
} void loop()
{
   if (digitalRead(pin) == 0)  // if the sensor's signal is zero,
i.e. the pendulum is in front of the sensor (in equilibrium
position)
   {
     if (most_kell_merni)       // the sensor's signal has to be
read (read the time) now, then
       {
         ido = micros();   // read the current value of time and
put it into the variable ido
         most_kell_merni = false;  delay(1);    // the value of
most_kell_merni is set to false, so that not to measure the
pendulum backwards
         if (i%20 == 0)  // measurement of 10T!
                     // the period T should be calculated in
           every second pass of the pendulum, but in the case
           of 10T this means every twentieth
          {
             delta_t = ido - elozo_ido;  // calculate the elapsed
time since the previous read, delta_t denotes here 10T
             elozo_ido = ido;           // let the value of the
elozo_ido be the time value just read – a new measurement
starts here
             Serial.println(delta_t);  // the time is written in
microseconds to the serial monitor  --> later convert to
seconds in a spreadsheet
           }
         i+=1;   // increment the counter
       }
   }
 else
   {
     most_kell_merni = true; delay(1);    // the value of
most_kell_merni is reset to true
   }
}
```

Fig. 8. shows an example of the results of our measurements to compare the motion of simple- and cycloidal pendulums, started with the same initial amplitudes.

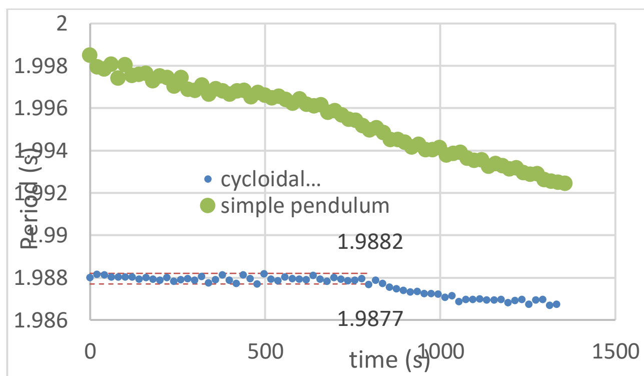

FIGURE 1. Period as a function of time for 98 cm long simple gravity pendulum and a cycloidal pendulum, started with the same initial amplitudes of 15 cm (α≈9º). A statistical standard deviation of 0.5 ms can be read from the figure.

It can be seen that while the period of the simple gavity pendulum decreases continuously, the period of the cycloidal pendulum of the same length is constant for approx. 830 seconds (13-14 minutes) and decreases by only 0,001 second in the following 8 minutes. Hence, we can state that if a cycloidal suspension of appropriate shape is obtained for a given pendulum length (or a suitable pendulum length can be found found for a given cycloid), we actually obtain a pendulum with a constant, amplitude- independent periode T.

It is worth comparing the values of g obtained from the steady-state sections of the above (non-manual) measurements for mathematical pendulums, taking into account the first 7 terms of (4), which are given in the table below (the estimated error of pendulum length measurement is ±1 mm):

	g (m/s^2)
simple grav. pendulum, l=98±0,1 cm, α=2º (Fig. 1)	**9,750**±0,001
simple grav. pendulum, l=573±0,1 cm, α=1º (Fig. 2)	**9,788**±0,001
cycloidal pendulum on the „needle-suspension" of Fig.4, l=99±0,1 cm, α=2º (Fig. 5)	**9,849**±0,001
cycloidal pendulum, l=98±0,1 cm, α=2º (Fig. 7)	**9,789**±0,001

These g values approximate well the "Budapest g" in Hungary (which is presumably close to g at Nyíregyháza). The development of the pendulum (from the simple gravity pendulum to the cycloidal one) gives g values permanently approaching to 9.81m/s^2. The value of g obtained with the cycloidal pendulum on the „needle-suspension" shows a slightly different value: it can be assumed that the needles could have affected the movement of the pendulum, as the pendulum often deviated from its plane of oscillation, while this was not the case using a 3D-printed cycloidal suspension. We can also declare that not only was the amplitude independence extended to larger amplitudes with the above improvements, but we also achieved that more accurate g values could be obtained with an even shorter pendulum using the cycloidal shaped suspension.

IV. THE CYCLOIDAL PENDULUM IN SCHOOL PHYSICS

Study of the topic can be a project work of talented high school students who are interested in specialization, but it is also worth sharing the conclusions for larger classes within normal physics classes. The difficulties in mathematics mentioned above can easily be overcome with a short roaming in the history of science and learning about the properties of the cyclois. Drawings and programs needed to make the suspension can be found in the References below. Students can realize the curiosity of this topic as we plot time as a function of time in these measurements. Newertheless, analysis of these plots is definitely useful in developing their abstract thinking.

Discussion of the topic is obviously important because students have already studied about pendulum motion more times in school (in class 7 and class 10) and it is also a task in the intermediate level school leaving exam: the gravitational acceleration has to be deetrmined using a simple gravity pendulum. It is a relatively easy to carry out measurement, that provides a kind of connection between knowledges learned at different times: the free fall and periodic movements (7). Historical aspects can also be mentioned when performing the measurement: the error of the fall time in free fall measurements can be very large, since the human reaction time is comparable to the fall time at low heights. Pendulum measurements (measuring the time of 10 periods), however, have already much smaller errors. During the oral exam, students have to emphasize that only in case of small amplitudes – less than approx. 5º – could be got a more accurate value, in case of larger amplitudes the period varies, and must explain the possible reasons of the difference between the expected value and the measured values, as well.

$$g = 4\pi^2 \cdot \frac{l}{T^2}. \quad (7)$$

Before carrying out the IR-sensor measurements, it is worth starting with some more simple measurements: to measure the period of a longer pendulum at different amplitudes (at least three measurements for each amplitude). The analysis of the measurement data will reveal the amplitude dependence of the simple gravity pendulum. Although not specifically related to the topic, it is worth to solve the following task [10] so that to go on, and see the video [11] for solution:

An ideal gravity pendulum is deflected 90 degrees from its equilibrium position and then released. As the pendulum passes through the vertical position, the cord collides with a needle. The question is, where the needle should be placed so that the trajectory of the pendulum's bob is a circle?

This example highlights the role of obstacles to solve the original problem. Let students measure the period again (eg. at an amplitude of 30 degrees) with well-positioned needles. Experiments will show a decreasing period. Be the next task to find the description of the general pendulum motion and then analyze and plot it! Based on the graph, the shortening of the cord and the posititions of the needles are determined step by step! Perform period measurements again at different amplitudes. Having analyzed the errors, students get acquainted with the curve of the general solution: the cycloid and its scientific background. Plot the students the cycloid in different ways: drawing on a board, drawing in a booklet, and using a computer, too! If possible, print it using a 3D printer or use a template to position the obstacles (needles). Measure again and compare the results obtained with and without the obstacles, then conclude! If possible, perform the IR-sensor measurements with teacher's control. Compare with stopwatch measurements!

Students make presentations on the topic and present it first to each other and then to classmates. It should be presented as speciality, but it should be emphasized that this pendulum indeed has a period that can be calculated from (1) even at larger amplitudes, and this is not true for the pendulum bob moving on circular trajectory, only at small amplitudes.

A spectacular effect can be obtained by starting two pendulums opposite each other from the same suspension (so that they do not collide, see Fig. 6), with different initial amplitudes. They then reach their vertical position at the same time in each periode due to their tautochron properties.

A pretty interesting "physics game" can be made based on the cycloidal pendulum. If we draw the trajectory of the moving pendulum bob, we can make a brachistochron slope of a cycloydal arc. If the slope is suitably designed, let a ball move on it at the same time by starting the pendulum body! We would find that the ball and the pendulum bob move with the same period. The movement of the ball will obviously dampen much quicker due to the greater damping rate of the friction, but the continuously decreasing displacements of the ball will be in sync with the pendulum due to the tautochron property of the slope.

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