Motion of connected wheels

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Abstract

We present three examples of the motion of connected two rolling objects as students' practice exercises: (i) connected by a massless string moving on a flat floor, (ii) connected by a massive rod moving on an inclined plane, and (iii) connected by a massless belt moving on a flat floor. To write down and solve the equations of motion, one has to take the string and belt tension and the force from the rod into account. We also discuss the work-energy relation of each rolling object. This topic will help students' understanding of the rolling motion.

Keywords: rolling motion, moment of inertia, rolling friction, work-energy relation.

Resumen

Presentamos tres ejemplos del movimiento de dos objetos rodantes conectados como ejercicios de práctica de los estudiantes: (i) conectados por una cuerda sin masa que se mueve sobre un piso plano, (ii) conectados por una barra masiva que se mueve sobre un plano inclinado, y (iii) conectados por una correa sin masa que se mueve sobre un piso plano. Para escribir y resolver las ecuaciones de movimiento, hay que tener en cuenta la tensión de la cuerda y la correa y la fuerza de la varilla. También discutimos la relación trabajo-energía de cada objeto rodante. Este tema ayudará a los estudiantes a comprender el movimiento de balanceo.

Palabras clave: movimiento de rodadura, momento de inercia, fricción de rodadura, relación trabajo-energía.

I. INTRODUCTION

While there have been many studies of the rolling motion from a pedagogical point of view, most of the cases discuss the motion of a single rolling object. As is the case with the translational motion, a more advanced learning step in the rolling motion would be to learn about the motion of connected objects [1, 2, 3, 4, 5]. Here we present three examples of the motion of two rolling objects: (i) connected by a massless string moving on a flat floor, (ii) connected by a massive rod moving on an inclined plane, and (iii) connected by a massless belt moving on a flat floor. One can write down the equations of motion for translational and rotational motion of each wheel including forces from the string, the rod, and the belt. These forces are the internal forces of the whole system.

The wheel slightly deforms because of its weight, which is the origin of the rolling friction. The internal forces are a function of the deformation parameter. We also discuss the work-energy relation of each wheel and show the role of the internal forces. These will be students' practice exercises in university classes.

II. ANALYSIS

A. Connected by massless string

As the first example, let us consider the motion of two equivalent wheels with mass m and radius r connected by a massless string as shown in FIGURE 1. Suppose that the wheels are rolling on a flat floor without slipping, then its translational and angular acceleration, a and α, are related by $a = r\alpha$. The moment of inertia of each wheel around the axle is $I = \beta m r^2$, where β is the mass distribution factor ($\beta = 0.5$ for a disk and $\beta = 1$ for a ring). If the system is accelerated by the torque τ_1 at the front wheel, the static frictional force $F_1 (F_2)$ at the front (rear) wheel is in the right (left) direction. The string tension T is acting on each wheel as depicted in the figure. Here and hereafter, the subscripts 1 and 2 represent the physical quantities for the front and rear wheel, respectively.

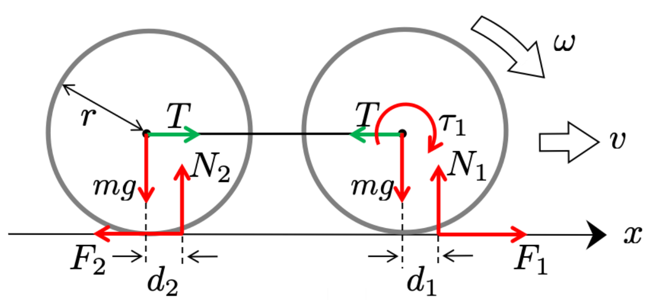

FIGURE 1. Two equivalent wheels connected by a massless string are accelerating on a flat floor because of the torque τ_1 at the front wheel.

The wheels roll according to the equations of motion for translation

$$\text{front} : ma = F_1 - T, \quad (1)$$

$$\text{rear}\ : ma = -F_2 + T, \tag{2}$$

and for rotation

$$\text{front} : I\alpha = \tau_1 - F_1 r - N_1 d_1, \tag{3}$$

$$\text{rear}\ : I\alpha = F_2 r - N_2 d_2, \tag{4}$$

where $N_{1,2} = mg$ is the normal force whose line of action is shifted by a distance $d_{1,2} = \delta r$, with the small dimensionless parameter $\delta\,(|\delta| \ll 1)$, because of the deformation of the wheels [6].

From Eqs. (1) to (4), one easily finds the acceleration

$$a = \frac{1}{1+\beta}\left(\frac{\tau_1}{2mr} - \delta g\right), \tag{5}$$

which describes that the wheels are accelerated by the torque τ_1 and decelerated due to the deformation δ, as expected. The frictional forces $F_{1,2}$ are given by

$$F_1 = \frac{1}{1+\beta}\left[\left(1+\frac{\beta}{2}\right)\frac{\tau_1}{r} - \delta mg\right], \tag{6}$$

$$F_2 = \frac{1}{1+\beta}\left(\frac{\beta}{2}\frac{\tau_1}{r} + \delta mg\right), \tag{7}$$

and T is thus

$$T = \frac{F_1 + F_2}{2} = \frac{\tau_1}{2r}. \tag{8}$$

When $\tau_1/r = 2\delta mg$, the acceleration vanishes, $a = 0$, corresponding to the motion with a constant velocity. In this case, $F_1 = F_2 = T = \delta mg$ is also satisfied.

The mechanical energy of each wheel is generated by work done by external forces. As for the rear wheel with the initial condition $v_0 = 0$ ($\omega_0 = 0$), the mechanical energy of the initial state is $E_{2(0)} = 0$ and of the state at an arbitrary time is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(1+\beta)mv^2. \tag{9}$$

During the period that the wheel moves at a distance $L = 2\pi r N$ (N is the number of rotations of the wheel), the static frictional force F_2 decreases the translational kinetic energy by $W_{F_2}^{(k)} = -F_2\, L$ and increases the rotational kinetic energy by $W_{F_2}^{(r)} = F_2\, r \times 2\pi N$. Therefore, the net work done by F_2 vanishes, $W_{F_2} = W_{F_2}^{(k)} + W_{F_2}^{(r)} = 0$ [7,8]. For the rear wheel, the works W_T done by T and energy dissipation W_{N_2} by the deformation of the wheel, are given by

$$W_T = TL = \frac{\tau_1 L}{2r}, \tag{10}$$

$$W_{N_2} = -N_2 \times 2\pi d_2 N = -\delta mgL. \tag{11}$$

Thus, the total energy loss $W_2 = W_T + W_{N_2}$ [7] is

$$W_2 = \left(\frac{\tau_1}{2r} - \delta mg\right)L = (1+\beta)maL, \tag{12}$$

where Eq. (5) has been imposed in the last equality. Eq. (12) coincides with Eq. (9) because of the relation $v^2 = 2aL$. One finds that the work-energy relation $E_{2(0)} + W_2 = E_2$ is satisfied. As for the front wheel, $E_{1(0)} = 0$ and $E_1 = E_2$. The work is $W_1 = -W_T + W_{N_1} + W_{\tau_1}$, where $W_{\tau_1} = \tau_1 \times 2\pi N$ is the work done by τ_1. One can easily confirm that $E_{1(0)} + W_1 = E_1$ is satisfied in a similar fashion. For the whole system, the total energy loss is $W = W_1 + W_2 = W_{N_1} + W_{N_2} + W_{\tau_1}$, in which W_T done by an internal force T is eliminated. In this way, one can verify the work-energy relation for each wheel by taking the internal force T and the torque of $N_{1,2}$ [7].

B. Connected by massive rod on inclined plane

The second example is two equivalent wheels connected by a massive rod going down on an inclined plane due to gravity. The rod has mass M and length ℓ, and the angle of the inclined plane is θ to the horizontal. In this case, the system can be separated into three objects as shown in FIGURE 2.

The forces T_1 and T_2 exerted between the rod and the wheels are different from each other because the rod has mass, unlike the previous case. The green and blue arrows in FIGURE 2 represent the action-reaction pairs exerting on the rod and the wheels. We take the z-axis fixed at the rod, and the center of mass of the rod is at $= Z$.

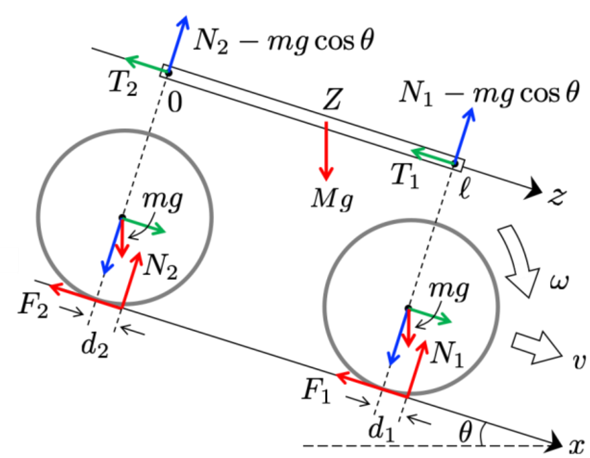

FIGURE 2. The system of two equivalent wheels connected by a massive rod naturally goes down on an inclined plane due to gravity. The green and blue arrows at the rod and the wheels are the action-reaction forces.

The system moves according to the equations of motion for translation

$$\text{front} : ma = -F_1 + T_1 + mg\sin\theta, \tag{13}$$

$$\text{rear}\ : ma = -F_2 + T_2 + mg\sin\theta, \tag{14}$$

$$\text{rod} \;\;: Ma = -T_1 - T_2 + Mg\sin\theta, \qquad (15)$$

and for rotation

$$\text{front}: I\alpha = F_1 r - N_1 d_1, \qquad (16)$$

$$\text{rear} \;\;: I\alpha = F_2 r - N_2 d_2. \qquad (17)$$

The displacement $d_{1,2}$ is assumed to be proportional to $N_{1,2}$ as

$$d_{1,2} = \delta r \frac{2N_{1,2}}{(M+2m)g}. \qquad (18)$$

The equilibrium condition for the rod in the normal direction is

$$N_1 + N_2 = (M+2m)g\cos\theta. \qquad (19)$$

Since the rod never rotates during the motion, the equilibrium condition of the torque around the axle of the rear wheel

$$0 = MgZ\cos\theta - (N_1 - mg\cos\theta)\ell, \qquad (20)$$

has to be satisfied. Therefore, the acceleration is

$$a = \frac{g}{1+2(1+\beta)\tilde{m}}\left[(1+2\tilde{m})\sin\theta - \frac{2\delta}{1+2\tilde{m}}\cos^2\theta\left(2\tilde{Z}^2 - 2\tilde{Z} + 1 + 2\tilde{m} + 2\tilde{m}^2\right)\right], (21)$$

where $\tilde{m} = m/M$ and $\tilde{Z} = Z/\ell$. Notice that the acceleration has its maximal value $a_{\max}$ at $\tilde{Z} = 1/2$. In this case, one finds

$$a_{\max} = \frac{(1+2\tilde{m})g}{1+2(1+\beta)\tilde{m}}(\sin\theta - \delta\cos^2\theta), \qquad (22)$$

and

$$F_1 = F_2 = (1+2\tilde{m})T_1 = (1+2\tilde{m})T_2 = \frac{(1+2\tilde{m})Mg}{1+2(1+\beta)\tilde{m}}\left[\beta\tilde{m}\sin\theta + \frac{\delta(1+2\tilde{m})}{2}\cos^2\theta\right]. (23)$$

As a consistency check, one can easily find that Eq. (22) reduces to Eq. (5) with $\tau_1 = 0$ for $M = 0$ ($\tilde{m} \to \infty$) and $\theta = 0$, as expected. Since the forces $F_{1,2}$ and $T_{1,2}$ have more complicated forms for a general $\tilde{Z}$, it would not be very educational to derive the exact forms for introductory students. Instead, the optimal problem to find $\tilde{Z} = 1/2$ from Eq. (21) would be a good exercise for students. See [9] for details.

As for the work-energy relation of the front wheel for $\tilde{Z} = 1/2$, $E_{1(0)} = 0$ when taking the origin of the potential energy at the initial position, and

$$E_1 = \frac{1}{2}(1+\beta)mv^2 - mgL\sin\theta, \qquad (24)$$

after going down the distance L along the x-axis. During its motion, the work done by the external forces is

$$\begin{aligned} W_1 &= W_{T_1} + W_{N_1} \\ &= T_1 L - N_1 \times 2\pi d_1 N, \\ &= \frac{T_1 + T_2}{2}L - \delta\frac{2N_1^2}{(M+2m)g}L, \\ &= (1+\beta)ma_{\max}L - mgL\sin\theta, \end{aligned} \qquad (25)$$

where Eqs. (15) and (20) have been imposed. In this way, one finds that the work-energy relation $E_{1(0)} + W_1 = E_1$ is satisfied, and it is similar for the rear wheel.

C. Spools connected by massless belt

Finally, consider two equivalent spools of mass m and radius R that are connected by a massless belt wrapped around axles of radius r, as shown in FIGURE 3. The spools are moving with a constant velocity $v = R\omega$ because of the torque τ_1 at the axles of the front spool. The belt tensions are T_u and T_d as depicted in the figure.

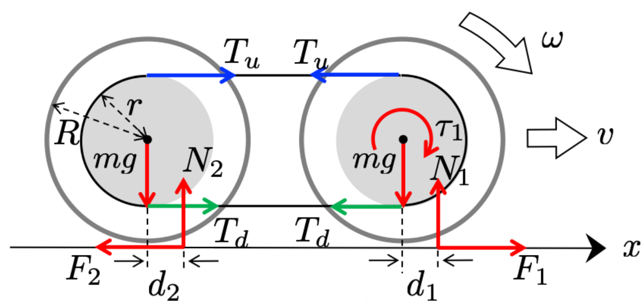

FIGURE 3. The system of two equivalent spools of radius R connected by a massless belt around axles of radius r on a flat floor. The belt pulls the spools by the forces depicted by blue and green arrows.

The equations of motion for translation are

$$\text{front}: ma = F_1 - T_u - T_d, \qquad (26)$$

$$\text{rear} \;\;: ma = -F_2 + T_u + T_d. \qquad (27)$$

We assume that T_u and T_d are related by the capstan equation (belt friction equation)

$$T_u = T_d e^{\mu\pi}, \qquad (28)$$

where μ is the coefficient of static friction between the belt and the spool. Since $T_u > T_d$ because of $e^{\mu\pi} > 1$, it shows that the lower belt will be looser a little more than the upper one. See Appendix for the derivation of Eq. (28).

The equations of motion for rotation are

$$\text{front}: I\alpha = \tau_1 - F_1 R - N_1 d_1 - r(T_u - T_d), \quad (29)$$

$$\text{rear}: I\alpha = F_2 R - N_2 d_2 + r(T_u - T_d), \quad (30)$$

with $N_{1,2} = mg$ and $d_{1,2} = \delta R$.

The constant velocity conditions $a = \alpha = 0$ give

$$F_1 = F_2 = T_u(1 + e^{-\mu\pi}), \quad (31)$$

from Eqs. (26)~(28). Moreover, we obtain

$$\delta = \frac{\tau_1}{2mgR}, \quad (32)$$

and

$$T_u = \frac{\tau_1/2}{R(1 + e^{-\mu\pi}) + r(1 - e^{-\mu\pi})}, \quad (33)$$

from Eqs. (29) and (30). If the spools consist of a perfectly rigid material, that is, if $\delta = 0$, $\tau_1 = 0$ is required to keep the constant motion. This indicates that a non-zero torque τ_1 is needed for a constant motion for realistic spools, as expected.

Since the kinetic energy of the system given by Eq. (9) is constant in our present case, the total work done by the forces vanishes. Again, we separately calculate the energy loss for the front and the rear spool, W_1 and W_2.

For the front spool, since the static frictional force F_1 does no net work as discussed above, $W_1 = -W_{T_u} - W_{T_d} + W_{N_1} + W_{\tau_1}$. When the spools move the distance $L = 2\pi RN$, the displacement of the point of application of T_u is

$$L + 2\pi rN = L\left(1 + \frac{r}{R}\right). \quad (34)$$

Therefore

$$W_{T_u} = T_u L\left(1 + \frac{r}{R}\right), \quad (35)$$

and in a similar fashion,

$$W_{T_d} = T_d L\left(1 - \frac{r}{R}\right), \quad (36)$$

and finally, W_{N_1} is the same as W_{N_2} given by Eq. (11). From Eqs. (35), (36), and (11), one obtains

$$W_1 = -T_u L\left(1 + \frac{r}{R}\right) - T_d L\left(1 - \frac{r}{R}\right) - \delta mgL + 2\pi N\tau_1, \quad (37)$$

where the first three terms decrease the translational and rotational kinetic energy of the spool, while the last term increases. In our present case, one finds that

$$W_1 = -\frac{T_u L}{R}[R(1 + e^{-\mu\pi}) + r(1 - e^{-\mu\pi})] - \delta mgL + 2\pi N\tau_1 = 0, \quad (38)$$

by using Eqs. (32) and (33). Therefore, the front spool has no energy loss. One can prove $W_2 = 0$ in a similar fashion.

III. SUMMARY

To summarize, we have presented three examples of the motion of connected wheels or spools as students' practice exercises to deepen students' understanding of the rolling motion. Extending the problem to that with the torque τ_2 at the rear wheel like a bicycle, or wheels of different mass such as m_1 and m_2 will be straightforward.

APPENDIX

Here we derive the capstan equation Eq. (28). A cylinder of radius r is rotating counterclockwise around the axle with a constant angular velocity ω, because of the friction between the belt and the cylinder as shown in FIGURE 4. The tensions of the belt are T_1 and T_2, and the contact angle is ϕ. Since the situation of FIGURE 4 corresponds to the rear spool of FIGURE 3, we will replace $T_2 \to T_u$ and $T_1 \to T_d$ after the derivation of the capstan equation.

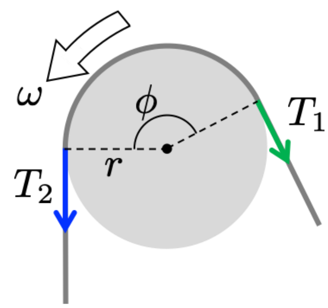

FIGURE 4. A cylinder of radius r is rotating with a constant angular velocity ω around the axle because of the friction between the belt and the cylinder.

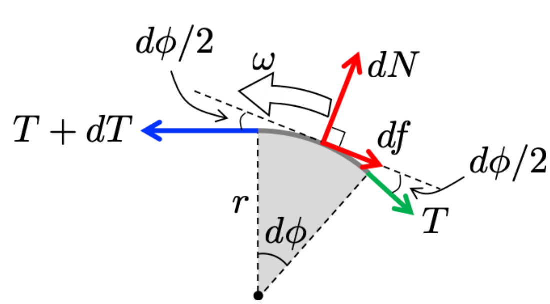

FIGURE 5. The forces exerted on the belt of a circular sector of small angle $d\phi$.

FIGURE 5 represents the forces exerted on the belt of a circular sector of small angle $d\phi$. The normal force dN that the belt receives from the cylinder is balanced with the tensions T and $T + dT$ as

$$\begin{aligned} dN &= T\sin\left(\frac{d\phi}{2}\right) + (T + dT)\sin\left(\frac{d\phi}{2}\right) \\ &\simeq T\frac{d\phi}{2} + (T + dT)\frac{d\phi}{2} \\ &\simeq T d\phi, \end{aligned} \quad (A.1)$$

where $\sin d\phi \simeq d\phi$ for $|d\phi| \ll 1$ and $d\phi dT \simeq 0$ have been applied. As for the tangential direction, since the belt tension is balanced with a small element df of the frictional force as

$$(T + dT)\cos\left(\frac{d\phi}{2}\right) = df + T\cos\left(\frac{d\phi}{2}\right),$$

one obtains

$$df \simeq dT, \quad (A.2)$$

where $\cos d\phi \simeq 1$. Moreover, when the frictional force has its maximal value,

$$df \simeq \mu dN, \quad (A.3)$$

is satisfied, where μ is the static frictional coefficient.

From Eqs. (A.1) ~ (A.3), one obtains

$$\frac{dT}{T} = \mu d\phi,$$

and its integration

$$\int_{T_1}^{T_2} \frac{1}{T} dT = \int_0^{\phi} \mu d\phi$$

gives

$$\log\frac{T_2}{T_1} = \mu\phi, \quad (A.4)$$

or

$$T_2 = T_1 e^{\mu\phi}. \quad (A.5)$$

When $\phi = \pi$, Eq. (A.5) coincides with Eq. (28) by the replacement mentioned above.

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