Kinematics of a projectile motion and the hidden curves of aparabolic trajectory

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Abstract
We construct the parabolic trajectory of a projectile from the hidden curves associated with this motion. We show from these auxiliary curves how to derive the kinematic equations which describe the observed curve. The parabolic trajectory as a composition of a straight-line uniform motion and a free-fall one is also shown.

Keywords: kinematics; motion on a plane; projectile motion.

Resumen
Construimos la trayectoria parabólica de un proyectil a partir de las curvas ocultas asociadas con este movimiento. A partir de estas curvas auxiliares mostramos cómo derivar las ecuaciones cinemáticas que describen la curva observada. También se muestra la trayectoria parabólica como una composición de un movimiento uniforme rectilíneo y uno de caída libre.

Palabras clave: cinemática; movimiento en un avión; movimiento de proyectiles.

I. INTRODUCTION

The theory of projectiles moving in a uniform gravitational field such as the one we find near the surface of the Earth is well known. The projectile is launched with velocity $\mathbf{v}_0$, in a direction determined by the launching angle θ_0 with respect to the horizontal axis. The origin of the motion is at the point *O*, the origin of coordinates, and during its flight the projectile traces a parabola on the vertical *xy*-plane. This resulting parabolic trajectory of the projectile can be interpreted as the superposition of a uniform linear motion along the direction θ_0, and a downward motion due to the uniform acceleration of gravity, as shown in Figure 1. This composition of two independent motions was first recognised by Galileo [1, 2]. Torricelli, Galileo's collaborator at the time of his death, developed this concept, [3, 4] and all textbooks discuss its well-known features [5, 7, 6]. Some interesting features of the parabolic motion, however, are absent in modern textbook treatments and have been rediscovered recently. For instance, as illustrated in Figure 2, the locus of the vertex and focus of all possible parabolasfor a given set of initial conditions are associated with a circle and an ellipse, respectively [8, 9, 10].

In this note, we point out another hidden circle associated with the parabolic trajectory, and show that all these curves are related to the geometric construction of the parabola with the standards kinematic parameters of the parabolic motion. This process of construction of the parabola reveals the parabolic trajectory as a composition of two motions: a straight-lineuniform motion and a free-fall one.

II. GRAPHICAL CONSTRUCTION OF A PARABOLA

To construct graphically an arc of parabola, we proceed in the following way, as illustrated inFigure 1. Firstly, on the *xy*-plane, let an arc with centre at $(0, H)$ and diameter $2H$ cut a line *ON* whose origin is at *O* and makes an angle θ_0 with respect to the *x*-axis. Secondly, divide the line *ON* into *m* even equal parts and mark on it N_i points with $i = 2, 3, \ldots, m - 1$. Fromthese points trace perpendiculars to the *x*-axis. These lines cut the segment $a/2$ in *m* equal parts. The vertical line from *N* cuts the *x*-axis at $a/2$ and has the length $2h$. The choice of this notation will be clear later. This particular line defines the symmetry axis of the parabola. The point $(a/2, h)$ on this line localises the point *V*, the vertex of the parabola. Finally, cut the half inferior part of this line, also of length *h*, into 2^m parts; consider only the marks associated tothe first, fourth, . . ., and $(2m + 1)$-th part; and trace horizontal lines from these marks. The $m - 1$ points P_i, between the points *O* and *V*, generated by the intersection of the vertical and horizontal lines are on the parabola. To complete the curve for *x*-coordinates between *a/*2 and *a*, we extend the length of the line *ON* to twice of its original value, beyond the point *N* , divide it into the same equal *m* even parts, and proceed as before.

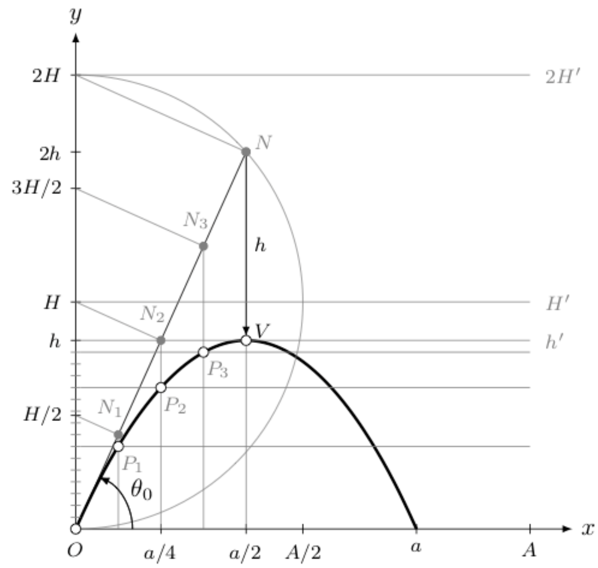

FIGURE 1. Graphically constructed parabola and projectile motion. This trajectory results from a composition of two motions: one, at constant v_0, along the line of shot, from point *O* to the point *N*; and another one, from point N_i to point P_i, vertically downwards.

Notice that, as indicated in Figure 2, the horizontal line starting at (0, *H*) corresponds to the directrix HH' of the parabola. The intersection of the directrix with the symmetry axis of the parabola defines the point *M*. The intersection of the line HN_2, in Figure 1, and the symmetry axis localises the point *F*, the focal point of the parabola. This result is also obtained by tracing an arc of circumference with centre at *O* and radius *H*. We observe that the location of the vertex *V* is at the arithmetic mean of the respective positions of the points *F* and *M*. As indicated by distances *OH* and *O*, the arc of parabola is the locus of all points P whose distance from the focal point F is equal to its distance from the directrix HH'.

To construct graphically the parabola, the partitioning of h into 2^m even parts is not mandatory [11, 12]. This division, however, is convenient in order to interpret the parabolic motion as a composition of two motions: one, at constant velocity $\overrightarrow{v_0}$, along the line of shot *ON*; and another one, vertically downwards under constant acceleration.

From Figure 2, we also note that, if v_0 is fixed, and θ_0 sweeps the interval $0 \leq \theta_0 \leq \pi/2$, the line *ON* generates parabolic trajectories whose foci F are on the arc of circumference of radius *H*, and respective vertexes *V* are on the arc of ellipse of major semiaxis *A/*2 and minor semiaxis *H/*2. The vertex V' corresponds to the parabola associated with the complementary angle $\theta_0{}' = \pi/2 - \theta_0$. Its focal point F' is on the intersection of symmetry axis of the parabola and the circle of radius *H* and centre *O* but with a the negative *y*-coordinate, and is not shown in Figure 2. These kinematic parameters of the parabolic motion of the projectile are discussed in the next section.

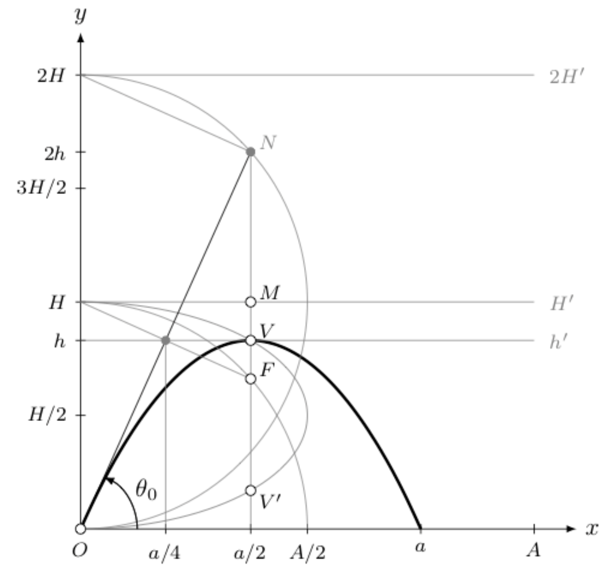

FIGURE 2. The parabolic trajectory, its directrix HH', focus *F*, and vertex *V*, for a given projectile shotted in direction θ_0 with speed v_0. Note the generated loci of different focus (an arc of circle) and vertexes (an arc of ellipse) for different elevation angles θ_0 of the segment *ON*.

III. KINEMATIC PARAMETERS OF THE MOVEMENT ON THE PARABOLA

Given the vertical semicircle with diameter 2*H* shown in Figure 1, suppose we launch a particle initially with velocity $\mathbf{v}_0$ from the origin *O* along the direction $\theta_0 = 0$. Let it move along the vertical line *OH* with negative constant acceleration *g*, until it attains null velocity at height *H*. It is convenient to define a characteristic length

$$H = \frac{{v_0}^2}{2g}, \tag{1}$$

and a characteristic time

$$t_H = \frac{v_0}{g}. \tag{2}$$

These two characteristic parameters are related by

$$2H = v_0 t_H, \tag{3}$$

Thus, if the particle could maintain its velocity equal to $\mathbf{v}_0$ along the line *OH*, during the same interval of time t_H, it would traverse a distance equal to 2*H*. The same reasoning can also be applied to a particle moving on the line *ON*, which is oriented by the angle θ_0 with respect to the *x*-axis. Notice that from Figure 1, that

$$ON = 2H\ \cos\left(\frac{\pi}{2} - \theta_0\right) = 2H\ \sin\theta_0. \tag{4}$$

Taking (1) into (4) we have

$$ON = v_0\left(\frac{v_0}{g}\sin\theta_0\right). \quad (5)$$

We can define a second characteristic time t_h by

$$t_H = \frac{v_0}{g}\sin\theta_0. \quad (6)$$

Therefore, while the particle maintains its velocity equal to $\mathbf{v}_0$ during the interval of time t_h, along the direction determined by θ_0, it moves a distance ON. Therefore, from Figure 1, the height $2h$ corresponds to

$$2h = 2H\sin^2\theta_0, \quad (7)$$

or, taking (6) into (7),

$$2h = (v_0\sin\theta_0)\,th. \quad (8)$$

Thus, the characteristic time t_h corresponds to the instant of time to which the vertical component of the velocity of the particle vanishes while it is moving under negative constant acceleration g, and has initial velocity $v_0\sin\theta_0$. This height h is then

$$h = \frac{{v_0}^2}{2g}\sin^2\theta_0. \quad (9)$$

and corresponds to the maximum height attained during its motion Notice from (9) that its greatest value is H, corresponding to $\theta_0 = \pi/2$. This result is illustrated in Figure 1. Moreover, we can see that for $\theta_0 = \pi/2$ the vertex V, and the focus F of the corresponding parabola, coincide at H.

Figure 1 also shows that the horizontal position $a/2$ is

$$\frac{a}{2} = 2H\cos\left(\frac{\pi}{2}-\theta_0\right)\sin\left(\frac{\pi}{2}-\theta_0\right),$$
$$= 2H\sin\theta_0\cos\theta_0, \quad (10)$$

or, replacing (6) in (10), we obtain

$$\frac{a}{2} = (v_0\cos\theta_0)\,th \quad (11)$$

Therefore, the distance a is associated to the range realised by the particle, corresponds to

$$a = \frac{v_0}{g}\sin 2\theta_0, \quad (12)$$

and is attained at the time $t_a = 2t_h$; equation (12) shows that the range a has a maximum value of $A = 2H$, for $\theta_0 = \pi/4$, the double of the height H. These characteristics are also indicated in Figure 2: when N coincides with M on the auxiliary circle of diameter $2H$, we must have $\theta_0 = \pi/4$, and the vertex V and the focus F of this parabola are at $(A/2, H/2)$ and $(A/2, 0)$, respectively.

We also remark, inspecting Figure 2, that $OF = H$, and the focus F is localised at

$$x_F = H\sin 2\theta_0, \quad (13)$$

$$y_F = H\cos 2\theta_0. \quad (14)$$

Then

$$x_F^2 + y_F^2 = H^2, \quad (15)$$

which represents a circle with centre at (0, 0) and radius H. Moreover, from the same figure, we also observe that the vertex V is localised at the arithmetic mean of the heights M and F.

Hence (7)

$$x_V = H\sin 2\theta_0, \quad (16)$$

$$y_V = \frac{1}{2}\left(H + H\cos 2\theta_0\right). \quad (17)$$

Then

$$\frac{(x_V)^2}{H^2} + \frac{(y_V - H/2)^2}{H^2/4} = 1, \quad (18)$$

or, observing that $A = 2H$,

$$\frac{(x_V)^2}{A^2/4} + \frac{(y_V - H/2)^2}{H^2/4} = 1. \quad (19)$$

This equation represents an ellipse with centre at $(0, H/2)$, minor semiaxis equal to $H/2$, and major semiaxis equal to $A/2$.

As indicated in Figure 3, let us name by y'-coordinate and x'-coordinate the symmetry axis of the parabola and the horizontal line, which cuts the height h, respectively, and take the vertex V as the origin of both coordinates in this new reference frame.

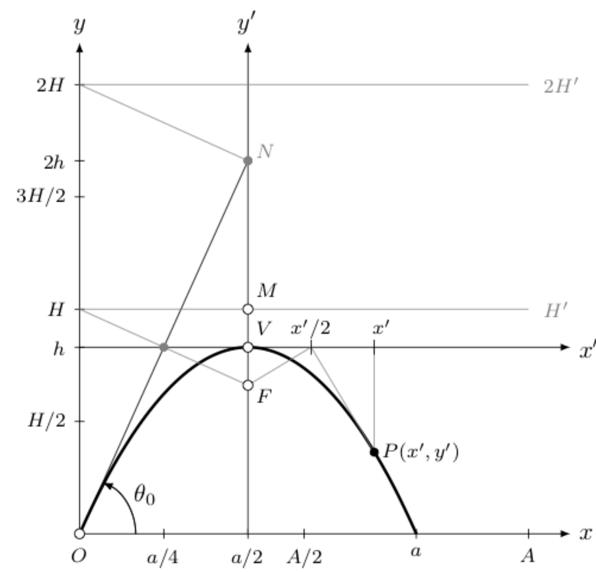

FIGURE 3. Parabola described in a new reference frame with $x'y'$-coordinates with origin at the vertex V. The segment $V\,F$ corresponds to the focal distance f of the parabola.

As indicated in Figure 3, a graphical construction of the parabola imposes that

$$\frac{y'}{x'/2}+\frac{x'/2}{f}, \qquad (20)$$

for each point on the parabolic arc, where f is the focal distance of the curve. The value of f is equal to the distance betweeen the heights of M and V. Then, from Figure 2, we get

$$f = H - H\sin^2\theta_0, \qquad (21)$$

or

$$f = H\cos^2\theta_0. \qquad (22)$$

Therefore, solving (20) for y' we obtain the equation

$$y' = -\frac{x'^2}{4f}. \qquad (23)$$

which describe the parabola in this new reference frame. This equation corresponds to the following coordinate transformation

$$x' = x - a/2, \qquad (24)$$

$$y' = y - h. \qquad (25)$$

We also remark that, from Figure 3, at $x' = a/2$ we have $y' = -h$. Therefore, taking these values into (23) we obtain

$$f = \frac{a^2}{16h}. \qquad (26)$$

Now, if we take (24), (25), and (26) into (23), the parabola will be described in the originalreference frame by

$$y = \left(\frac{4h}{a}\right)x - \left(\frac{4h}{a^2}\right)x^2. \qquad (27)$$

We note, from Figure 2, that

$$\frac{h}{a/4} = tan\,\theta_0. \qquad (28)$$

Taking (26) into (22) we obtain

$$\frac{a^2}{16h} = H\,cos^2\,\theta_0, \qquad (29)$$

or

$$\frac{4\text{h}}{a^2} = \frac{1}{H}(1+\tan^2\theta_0). \qquad (30)$$

Therefore,

$$y = \tan\theta_0\,x - \frac{1}{2\text{H}}\,(1+\tan^2\theta_0)\,x^2, \qquad (31)$$

or

$$y = x\tan\theta_0\left(1-\frac{x}{2H\sin 2\theta_0}\right). \qquad (32)$$

Equation (32) is most convenient if we wish to check out the expressions for the maximum height h and the maximum range a attained by the projectile. We also observe, from Figure 1, that the position x of the particle evolves uniformly, hence we write

$$x = (v_0 t)\cos\theta_0. \qquad (33)$$

Finally, taking (33) into (31) we have

$$x = v_0 t\cos\theta_0, \qquad (34)$$

$$y = v_0 t\sin\theta_0 - \frac{1}{2}gt^2, \qquad (35)$$

that is, the position P (x, y) of the particle on the parabolic curve in parametric representation.

IV. CONCLUSIONS

These and other interesting features related to the parabolic motion are not discussed in the modern textbooks that are used in high school and basic university physics. As long as the authors are aware of only in one dated textbook [5] and in references we can find material similar to the one discussed here. The authors believe that these results, simple from the point of view of the previous mathematical knowledge required from the students may awaken their curiosity and enliven the discussion of kinematics in the classroom.

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