

# Quantum Mechanics of the Solar System

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(Received 9 January 2014, accepted 16 June 2014)

## Abstract

According to the correspondence principle, as formulated by Bohr, both in the old and the modern quantum theory, the classical limit should be recovered for large values of the quantum numbers in any quantum system. However, this classical limit of quantum theory is not so straightforward as in the interface of other generalizations of classical mechanics and other domains. In particular, relativistic kinematics and mechanics reduce to Newtonian equations by simple algebra in the case of bodies moving with small velocities compared to the speed of light in vacuum. In this paper we consider the correspondence limit to the two-body problem in gravitational physics, the limit in which both the principal and the angular quantum numbers,  $N$ ,  $L$  are very large. In this limit, we compare with the classical elliptical orbits and we find that the macroscopic coherent quantum states correspond to the statistical average of every classical state compatible with conservation laws for the total energy and angular momentum. We also consider the perturbed Kepler problem with a central perturbation force proportional to the inverse of the cube of the distance to the central body. The exact solution for the quantum eigenstates shows that the first order perturbation to the energy eigenvalues are obtained classically as the temporal orbital average of the perturbation potential.

**Keywords:** Rydberg states, Kepler problem, Orbital perturbations.

## Resumen

De acuerdo con el principio de correspondencia, tal como fue formulada por Bohr, tanto en la vieja y en la moderna teoría cuántica, el límite clásico se debe recuperar para valores grandes de los números cuánticos en cualquier sistema cuántico. Sin embargo, este límite clásico de la teoría cuántica no es tan sencillo como en la interfaz de otras generalizaciones de la mecánica clásica y otros dominios. En particular, la cinemática relativista y la mecánica se reducen a las ecuaciones de Newton por álgebra simple, en el caso de los cuerpos que se mueven con velocidades pequeñas en comparación con la velocidad de la luz en el vacío. En este artículo se considera el límite de la correspondencia con el problema de los dos cuerpos de la física gravitacional, el límite en el que tanto el número cuántico principal como los números cuánticos angulares,  $N$ ,  $L$  son muy grandes. En este límite, se compara con las órbitas elípticas clásicas y se encontró que los estados cuánticos coherentes macroscópicos corresponden al promedio estadístico de cada estado clásico compatible con las leyes de conservación de la energía total y el momento angular. También se considera que el problema de la perturbación de Kepler con una fuerza de perturbación central proporcional a la inversa del cubo de la distancia al cuerpo central. La solución exacta para los estados cuánticos propios muestra que la perturbación de primer orden a los valores propios de energía es obtenida clásicamente como la media orbital temporal de la potencial perturbación.

**Palabras clave:** estados de Rydberg, problema de Kepler, perturbaciones orbitales.

**PACS:** 03.65Ge, 03.65.Ta, 45.20.-d

**ISSN 1870-9095**

## 1. INTRODUCTION

The discovery of new phenomena beyond the domain of validity of physical theories usually requires the formulation of new theories encompassing both the classical applications of the former theory and the novel ones. In a purely empirical way the new theory must reduce to the original by producing the same predictions in the restricted domain in which the old theory has proven right for a long time. This commonly, but not necessarily,

implies that, mathematically, these two different theories coincide in the smaller domain of the previous one.

However, this not means that the novel theory is conceptually equivalent because, despite their formal correspondence, both theories could base their predictions on very different standpoints. This evolution of physical theory has taken place many times in the history of science.

A well-known example is the case of special relativity and its reduction to classical mechanics in the limit of small velocities compared to the speed of light. In this limit,

Lorentz transformations become Galilean transformations and the total energy of a moving body reduce to the classical expression for the kinetic energy save for the rest energy term, the last one being a characteristic prediction of special relativity [1].

From a pedagogical point of view this is a very useful derivation because it allows us to gain confidence in the new theory by retrieving the well-known results of the previous theory subsumed into the most recent one. Hence, a correspondence principle appears in any scientific revolution or generalization between the modern theory and the previous one subsumed by it. The reasons for the existence of such a principle are clear: if the previous theory has been successful till the discovery of anomalies in another domain, it means that, in that domain at least, is formally correct. So, the new theory should reduce to it to avoid conflict with previous experiments.

In cases of radical conceptual departure from the classical theory, such as it has occurred with quantum mechanics, the application and interpretation of the correspondence principle is not an intuitively simple task.

Already for the old quantum theory and Bohr's model of the atom, the correspondence with the classical limit was proposed to be achieved for large values of the quantum numbers [2]. After the emergence of the modern Schrödinger's wave mechanics, and the equivalent abstract Heisenberg's matrix mechanics, Bohr's correspondence principle was adapted to them. However, the physical state in quantum mechanics is represented by a complex wave function interpreted in terms of the probability of finding the particle in a certain interval by means of Born's rule [3].

This is very different from the concepts of Newtonian mechanics in terms of particles following clearly-defined trajectories in space.

Another important step in the correspondence of quantum mechanics with classical physics was made by Ehrenfest in 1927 with the proof of the following theorem for a system with Hamiltonian  $H = p^2/(2m) + V(x)$ :

$$m \frac{d \langle x \rangle}{dt} = \langle p \rangle. \quad (1)$$

$$\frac{d \langle p \rangle}{dt} = - \left\langle \frac{dV(x)}{dx} \right\rangle. \quad (2)$$

Where  $\langle x \rangle$  and  $\langle p \rangle$  are the expectation values for the position and the momentum, respectively. This equation provides a correspondence with Newton's second law with the classical force as the expectation value of the potential gradient. The classical limit is assumed nowadays to emerge as a consequence of quantum decoherence, a process in which the superposition states loss the coherence of their phase angles by the irreversible interaction with the environment [4]. Consequently, pure quantum states cannot be held in macroscopic systems because the thermal interactions with the environment make them to lose their coherence. Nevertheless, the demand of the quantum computing project requires that coherence in quantum states should be maintained as long as possible because quantum computations rely on the superposition properties of

quantum states. As early as 1999, Nakamura *et al.* obtained 1  $\mu$ s coherence times for the two-level states of superconducting electrodes joined with Josephson junctions to a reservoir [5].

Another field where quantum mechanics has been applied to macroscopic systems is quantum cosmology. In 1967, the so-called Wheeler-DeWitt equation was proposed in the context of canonical quantum gravity as a model for the wave function of the Universe as a whole [6]. So, macroscopic quantum states are interesting in themselves both as a practical tool with possible engineering applications (as in the case of quantum computing). Also from a fundamental point of view, the analysis of macroscopic quantum states also shed light on the conceptual problems of quantum mechanics as a replacement of classical deterministic mechanics.

The study of the classical limit of the hydrogen atom was performed by Brown in 1973, who recovered the classical circular orbits for large  $n$  [7]. Experimental work on Rydberg wave-packets was even performed in the nineties of the past century and many similarities with classical behaviour was found [8, 9].

In this paper we revisit the two-body problem (pure Kepler problem and a problem with perturbations) with the tools of Schrödinger's wave mechanics and we discuss their interpretation in connection with classical physics. This example could be of real pedagogical interest for students because it covers subjects ranging from classical and quantum mechanics and the theory of perturbations and goes beyond the hackneyed quantum harmonic oscillator used in most texts.

The paper is organized as follows: In Section 2 we discuss the hydrogen atom solution of Schrödinger's equation in the limit of large principal,  $n$ , and angular,  $l$ , quantum numbers. We show that circular orbits correspond to  $l = n - 1$ . For  $l = n/2$  we discuss the statistical average correspondence among the available elliptical orbits, for fixed energy and angular momentum, and the quantum eigenstate. In Section 3 we analyze the effect of a perturbing potential  $\delta V(r) = -\alpha/(2r^2)$ , where  $\delta V(r)$  is the perturbation for unit mass and  $\alpha$  is a small constant. The relation among the solution of Schrödinger's equation for the eigenvalues of the energy and the classical theory of perturbations is also discussed. Finally, some remarks and conclusions are given in Section 4.

## II. THE QUANTUM KEPLER'S PROBLEM

We consider the solution of Schrödinger's equation for the gravitational potential  $V(r) = -G M m/r$ , where  $M$  is the mass of the central body,  $m$  is the mass of the orbiting body (assuming  $M \gg m$ ) and  $r$  is the distance among the bodies centers modelled as point-like or spherical. Mathematically speaking, the gravitational and electrostatic potential are the same. For this reason in Bohr and Sommerfeld models of the old quantum theory the hydrogen atom was described as a miniature solar system supplemented with quantization rules. In this paper we pursue the inverse analogy to describe a solar system as a macroscopic atom. Energy

eigenvalues of the two-body problem depend only on the principal quantum number,  $n$ , while the angular momentum eigenvalue is proportional to the angular quantum number,  $l$ :

$$E_n = -\frac{GMm}{2n^2 a_0} \quad (3)$$

$$J = l\hbar \quad (4)$$

Where  $a_0$  is the Bohr's radius of the system given by:

$$a_0 = \frac{\hbar^2}{GMm^2} \quad (5)$$

If we consider the values of the masses for the Sun,  $M = 1.989 \times 10^{30}$  kg, and the Earth,  $m = 5.972 \times 10^{24}$ , we get from Eq. (5) that the Bohr radius is  $a_0 = 9.27 \times 10^{-137}$  m, many orders of magnitude smaller than the Planck length.

As the mean distance of the orbiting particle to the force center is given by  $r_n = n^2 a_0$ . If we take  $r_n$  as the mean Earth-Sun distance, i. e.,  $r = 149.6 \times 10^6$  km we find  $n \approx 4 \times 10^{73}$  for the principal quantum number. These values are extremely large but, as we will soon see, even for  $n = 100$  the essential features of the classical limit are unveiled.

Another important relation for the orbital eccentricity,  $\varepsilon$ , is obtained from the classical relation of  $\varepsilon$  with the total energy and angular momentum of the planet [10] as follows:

$$\varepsilon = \sqrt{1 + \frac{2J^2 E / m}{(GMm)^2}} = \sqrt{1 - \frac{l^2}{n^2}} \quad (6)$$

Where we have used Eq. (3). Notice that Eq. (6) is the same result found in Sommerfeld's model of the atom. The eigenfunctions are derived exactly for the hydrogen's atom [11, 12] and they are usually expressed as the product of a radial part,  $R_{n,l}(r)$  and an angular part,  $F_{n,l,m}(\theta, \varphi)$ , i. e.,  $\Psi_{n,l,m} = R_{n,l}(r)F_{l,m}(\theta, \varphi)$  with:

$$R_{n,l}(r) = A_{n,l} \left( \frac{2r}{na_0} \right)^l e^{-r/(na_0)} L_{n-l-1}^{2l+1} \left( \frac{2r}{na_0} \right) \quad (7)$$

Where  $L_p^q(x)$  are the Laguerre polynomials of order  $p$ ,  $q$  [13] and the normalization coefficients are given by:

$$A_{n,l} = \left[ \frac{1}{2n} \left( \frac{2}{na_0} \right)^3 \frac{(n-l-1)!}{(n+l)!} \right]^{1/2} \quad (8)$$

The normalization condition for the radial function is:

$$\int_0^\infty |R_{n,l}(r)|^2 r^2 dr = 1 \quad (9)$$

The angular part is expressed in terms of the spherical harmonics for any central potential [11]:

$$F_{l,p}(\theta, \varphi) = Y_l^p(\theta, \varphi) = \sqrt{\frac{(2l+1)(l-p)!}{4\pi(l+p)!}} P_l^p(\cos \theta) e^{ip\varphi} \quad (10)$$

Where the last identity expresses the relation of the spherical harmonics with the associated Legendre polynomials,  $P_l^p(\cos \theta)$ , we have used  $p$  instead of  $m$  for the magnetic quantum number to avoid confusion with the mass. According to Born's principle, the modulus  $|Y_l^p(\theta, \varphi)|^2$  is proportional to the probability density for finding the particle in the solid angular interval  $d\theta d\varphi$ . As we are concerned with states analogous to macroscopic orbits we will restrict to the case with  $n, l$  large and a magnetic quantum number  $p = l$ . In Fig. 1 we show a plot of the angular factor of the probability distribution for  $n = l = p = 20$ . The distance to the origin is a measure of the probability density in the direction  $\theta, \varphi$  where  $\theta$  is the polar angle and  $\varphi$  is the azimuth. We notice that even for these relatively small values of the quantum numbers the confinement of the orbit to the plane perpendicular to the  $z$  axis is quite apparent. The reason for that confinement is clear from the identity:

$$Y_{l,l}(\theta, \varphi) = \sqrt{\frac{(2l+1)(2l)!}{4\pi}} \sin^l \theta \quad (11)$$

Which implies that the angular factor in the probability function decays very fast except for  $\theta = \pi/2$ , i. e., for the plane perpendicular to the  $z$  axis. So, we have shown that for  $l = p$  large the quantum state is restricted to a plane. In the following we will analyze the radial contribution to the probability function in order to elucidate the relation with classical orbits.

### A. Case $l=n-1$ and circular orbits

This case has been recently studied by Keepports [14] but we describe it here in some detail for completeness.

We consider the radial probability distribution,  $|rR_{n,l}(r)|^2$ , in the case of maximum angular quantum number,  $l = n - 1$ . This correspond to the probability of finding the particle in the interval  $r, r + dr$ . Moreover, as discussed before we know that this probability will only be non-negligible in the plane  $\theta = \pi/2$  as occurs in classical physics. From Eq. (7) and the properties of Laguerre polynomials [15] we find:

$$R_{n,n-1}(\rho) = \frac{1}{n^2(2n-1)!} \rho^{2n-2} e^{-2\rho/n} \quad (12)$$

Where  $\rho = r/a_0$  is the distance to the origin measured by taking Bohr's radius as our unit. The average distance to the origin is then given by:

$$\langle \rho \rangle = \int_0^\infty \rho^3 R_{n,n-1}(\rho) d\rho = n(n+1/2) \quad (13)$$

And, similarly, the variance is found to be

$$\sigma = \sqrt{\langle \rho^2 \rangle - \langle \rho \rangle^2} = n\sqrt{\frac{n-1}{2} + \frac{1}{4}}$$

And consequently,

$$\sigma / \langle \rho \rangle = 1 / \sqrt{2n+1}$$

And the radial probability distribution becomes sharper and sharper as n increases.

In Figure 2 a density plot of  $\rho^2 R_{n,l}^2(\rho)$  for  $n=50$  and  $l=49$  shows this behavior.

So, the quantum state becomes a circular orbit.

$$r^2 R_{n,n-1}(r)^2 \rightarrow \delta(r - n^2 a_0), \text{ as } n \rightarrow \infty. \quad (14)$$

Born's interpretation of the wave function indicates that the probability of finding the particle is uniform along the circular path. Classically we can interpret this result as a consequence of the uniform velocity with which the particle traverses its orbit. We can obtain the correspondent classical velocity from the quantum solution from the probability flux [11]:

$$\vec{j} = \frac{\hbar}{m} \text{Im} [\psi^* \vec{\nabla} \psi]. \quad (15)$$

Where Im denotes the imaginary part. Consequently, the only contribution to Eq. (15) comes from the imaginary term in the definition of the spherical harmonic as given in Eq. (10) for  $p=1$ . But,  $\partial \psi(n,l,1) / \partial \phi = i l \psi(n,l,1)$  and we have:

$$\vec{j} = \frac{\hbar}{m} \frac{l}{r \sin \theta} \psi \psi^* \phi = \frac{\hbar l}{m n^2 a_0} \phi + O(n^{-1}). \quad (16)$$

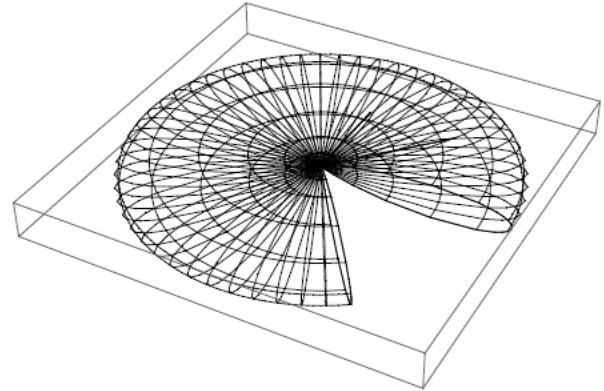
Where we have used Eq. (13) and we have also taken into account that  $\theta = \pi/2$  for non-negligible values of the distribution function.  $\phi$  is the unit azimuthal vector in the plane  $\theta = \pi/2$ , i. e., this is a unit vector tangential to circular orbits. A velocity vector can now be obtained as the quotient of the probability flux and the probability density as follows:

$$\vec{v} = \frac{\vec{j}}{\psi \psi^*} = \frac{\hbar l}{m n^2 a_0} \phi = \sqrt{\frac{GM}{a}} \phi. \quad (17)$$

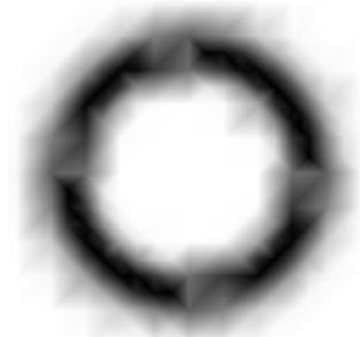
Notice that the last step is deduced from Eq. (5) and the classical angular momentum per unit mass for the circular orbit given by  $l \hbar / m = \sqrt{GMa}$ .

So, we finally deduce the velocity for the particle which traverses the classical circular orbit. In this interpretation, the quantum probability corresponds to the probability of finding, at a random time, the particle at a given point of the orbit. This temporal interpretation of quantum states in

the limit of large quantum numbers is usually applied in the case of the harmonic oscillator [12]. In the next section, we will show that we can understand elliptical orbits in the same way.



**FIGURE 1.** A slice of the three-dimensional plot for the square modulus of the spherical harmonic with  $l = p = 20$ . The distance of the surface to the origin is proportional to the probability density in that particular direction. The height of the parallelepiped is a ten per cent of the other two sides.



**FIGURE 2.** Density plot for the probability density with  $n=50$ ,  $l=p=49$ .

### B. Case $l < n-1$ and elliptical orbits

In this case we should find a relationship with classical elliptical orbits. In Fig. 3 we have plotted the probability  $P(r) = r^2 R_{n,l}(r)^2$  for finding the particle between two close spheres of radii  $r$  and  $r+d r$  in the case  $n=50$ ,  $l=n/2$ . For these values, the corresponding eccentricity is  $\varepsilon = \sqrt{3}/2$ . The classical orbits are given by [16]:

$$r = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon \cos \theta}. \quad (18)$$

A classical interpretation of the asymptotic quantum state for large  $n$ , and  $l < n - 1$  is obtained by performing a temporal average over all possible elliptic orbits compatible with conservation laws, i. e., all orbits with the same angular momentum and total energy. In Fig. 4 we plot ten orbits with eccentricity  $\varepsilon = 0.8$  whose semi-major axes are displaced an angle  $\pi/5$  with respect to those of their neighbouring orbits.

$$P(\rho) = \frac{1}{\pi} \frac{\rho}{2\rho - \rho^2 - 1 + \varepsilon^2} \quad (23)$$

Where  $\rho=r/a$  and the prefactor has been chosen in order to  $P(r)$  to be normalized in the interval  $0 \leq r < \infty$ . To compare with the quantum case we use  $n=1000$ ,  $l=500$  and the scaled radius  $\rho = r/(n^2 a_0)$ , i. e., in parallel with the circular orbit studied in the previous section the semi-major axis of the orbit correspond to  $a = n^2 a_0$ . To smooth the oscillations of the quantum probability distribution we perform an average over uniform intervals of width 0.1. The classical orbital average probability in Eq. (23) is compared with the averaged quantum result from Eq. (7) in Fig. 5. Notice that the orbital eccentricity of the classical orbits is given by Eq. 6 as  $\varepsilon = \sqrt{3}/2$  for  $l=n/2$ . The excluded classical regions are given by an inner circle whose radius is the perihelion distance to the central body,  $\rho_{\min} = 1 - \varepsilon \approx 0.134$ , while the maximum distance to the central body or aphelion is  $\rho_{\max} = 1 + \varepsilon \approx 1.866$ . We have shown that the probability distribution for a stationary quantum state of the hydrogen atom in the correspondence limit ( $n, l \gg 1$ ) can be interpreted as a temporal average over all possible trajectories sharing the same energy and angular momentum and with an eccentricity given by Eq. (6).

In parallel with the circular orbit case, we can calculate the quantum probability flux from Eq. (15) and we obtain the result:

$$\vec{j} = \frac{J}{r} \psi \psi^* \phi \quad (24)$$

Where  $J$  is the angular momentum per unit mass. This can be interpreted in terms of a transverse velocity:

$$\vec{v} = \frac{\vec{j}}{\psi \psi^*} = \frac{J}{r} \phi \quad (25)$$

This is a consequence of the orbital average over all elliptical orbits with the same energy and angular momentum. As displayed in Fig. 4, every point belongs to two elliptical orbits in such a way that the average of the radial motion is zero and only the transverse velocity appears after the average over orbits has been performed.

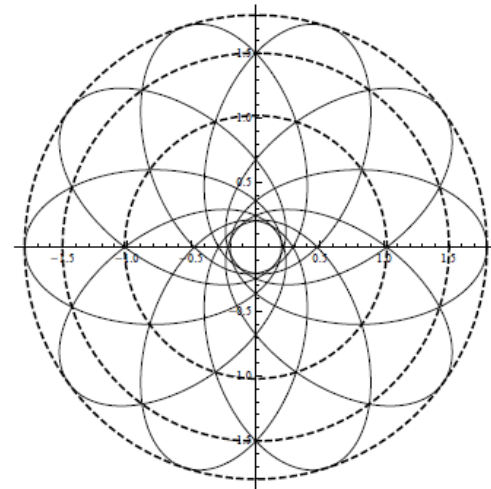


FIGURE 4. A set of ten elliptical orbits ( $\varepsilon=0.8$ ) sharing the same plane and uniformly rotated.

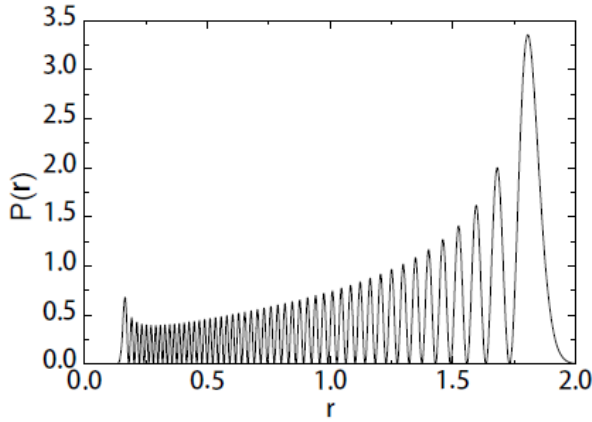


FIGURE 3. Radial probability density ( $n=50$ ,  $l=25$ ) for finding the orbiting particle.

We see from Fig. 4 that every point between the circles of radii between the perihelion,  $r_{\min} = a(1 - \varepsilon)$ , and the aphelion,  $r_{\max} = a(1 + \varepsilon)$ , belongs to the intersection of two rotated elliptical orbits. That point corresponds to opposite orbital angles or true anomalies:  $\theta$  and  $-\theta$ . So, the probability of finding the particle at a randomly chosen instant  $t$  should be proportional to the time spent by the particle in the angular interval  $\theta, \theta + d\theta$ :

$$P(\theta)d\theta = \frac{d\theta}{T\omega(\theta)} = \frac{d\theta}{TJ} r^2 \quad (19)$$

Where we have taken into account the relation of the angular velocity with the angular momentum (per unit mass) as given by Kepler's second law  $\omega(\theta) = J / r^2$ . We should also consider the expression of the angular momentum (per unit mass) in terms of the orbital geometric parameters, i. e., the semi-major axis and the orbital eccentricity [16]:

$$J = \sqrt{a(1 - \varepsilon^2)GM} \quad (20)$$

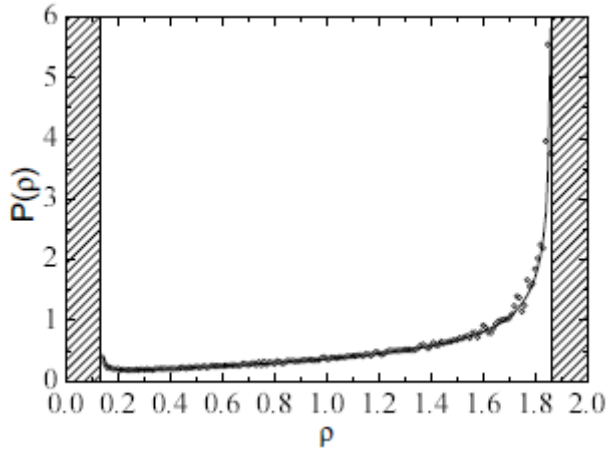
$M$  being the mass of the central body, as usual. Kepler's third law is also given as:

$$T = 2\pi a^{3/2} \mu^{-1/2} \quad (21)$$

By substitution of Eqs. (18), (20) and (21) into Eq. (19) we find:

$$P(\theta)d\theta = \frac{d\theta}{2\pi} \frac{(1 - \varepsilon^2)^{3/2}}{(1 + \varepsilon \cos \theta)^2} \quad (22)$$

It is more convenient for comparison with the quantum probability distribution to perform a variable change from  $\theta$  to  $r$ . So, we use Eq. (18) again to find:



**FIGURE 5.** The classical probability distribution for elliptical orbits (solid line) versus the smoothed quantum prediction for  $n=1000, l=500$  (circles). Dashed areas correspond to the excluded classical regions.

### III. CORRESPONDENCE FOR CLASSICAL ORBIT PERTUBATIONS

In this section, we consider a radial perturbation of the classical Newtonian potential and its effect on quantum stationary states in the correspondence limit. As the perturbation theory in quantum mechanics cannot be applied to very large values of the quantum numbers we will propose a problem with exact solution. We consider the Kepler problem with a  $1/r^3$  perturbation force term corresponding to the potential:

$$V(r) = -\frac{GMm}{r} - \frac{\alpha m}{2r^2}. \quad (26)$$

Where  $\alpha$  is a small quantity. The radial part of the wavefunction,  $u(r)=r R(r)$ , verifies the following equation:

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left\{ -\frac{GMm}{r} - \frac{\alpha m}{2r^2} + \frac{l(l+1)\hbar^2}{2mr^2} \right\} u(r) = Eu(r) \quad (27)$$

It is usually convenient to rewrite this radial equation in terms of non-dimensional parameters. We define a non-dimensional distance to the origin,  $\rho$ , and a parameter,  $\lambda$ , as follows:

$$\rho = \sqrt{-\frac{8mE}{\hbar^2}} r. \quad (28)$$

$$\lambda = \frac{GMm}{\hbar} \sqrt{\frac{m}{-2E}}. \quad (29)$$

This way we find that Eq. (27) can be casted into non-dimensional form as:

$$\rho \frac{d^2 u}{d\rho^2} + \left\{ \frac{\lambda}{\rho} - \frac{l(l+1)+\nu}{\rho^2} - \frac{1}{4} \right\} u(\rho) = 0. \quad (30)$$

Where  $\nu = \alpha m^2 / \hbar^2$ . Notice that Eq. (30) has the same structure that the radial equation for the standard hydrogen atom problem. The only difference arising from the extra term  $\nu$  in the denominator of the effective angular momentum potential. To solve this equation the standard approach resorts to the study of its behaviour in the limits of large and small  $\rho$ . For large  $\rho$  the terms inversely proportional to  $\rho$  and  $\rho^2$  are negligible and we have  $u(\rho) \propto e^{-\rho/2}$ . For small values of  $\rho$  the  $\rho^{-2}$  term dominates the behaviour and an algebraic solution  $u(\rho) = \rho^k$  is suggested. By direct substitution we find:

$$k(k-1) = l(l+1) + \nu. \quad (31)$$

Whose solution for  $k$  is:

$$k = \frac{1 + \sqrt{(2l+1)^2 + 4\nu}}{2} \approx l + 1 + \frac{\nu}{2l+1}. \quad (32)$$

Where the approximation is obtained on the assumption that  $\nu \ll 1$ . The standard change for the radial equation is then,  $u(\rho) = \rho^k f(\rho) e^{-\rho/2}$  and  $f(\rho)$  satisfies the second-order differential equation:

$$\rho \frac{d^2 f}{d\rho^2} + (2k - \rho) \frac{df}{d\rho} + (\lambda - k) f(\rho) = 0. \quad (33)$$

Whose solution is obtained by series expansion:

$$f(\rho) = \sum_{j=0}^{\infty} a_j \rho^j. \quad (34)$$

By inserting Eq. (34) into Eq. (33) we deduce that the coefficients  $a_j, j = 0, 1, 2, \dots$  are obtained through the following recurrence relation:

$$a_{j+1} = \frac{k + j - \lambda}{(j+1)(j+2k)}, \quad j \rightarrow \infty \quad (35)$$

It is well-known that  $a_j$  should be zero for some  $j$  because otherwise the series in Eq. (35) describes the exponential  $f(\rho) \propto e^\rho$  and the resulting eigenfunction  $u(\rho) \propto \rho^k e^{\rho/2}$  is not normalizable. If  $a_j = 0$  for some  $j$  the series in  $f(\rho)$  is finite and we have a polynomial solution. This condition will be written as:

$$\lambda = k + n - (l + 1). \quad (36)$$

This condition implies that Eq. (33) is verified by the associated Laguerre polynomials  $f(\rho) = L_{n-l-1}^{2k-1}(\rho)$ . From the definition of  $\lambda$  in Eq. (28) we find the perturbed energy levels:

$$E_{n,l} = -\frac{(GM)^2 m^3}{2\hbar^2} \frac{1}{\lambda^2}. \quad (37)$$

Notice that in the unperturbed case,  $\nu = 0$ ,  $\lambda = n$  and we recover Bohr's energy levels. The perturbed potential in Eq. (26) breaks angular momentum degeneracy and the energy levels now depend also on  $l$ . For small perturbations  $\nu \ll 1$  we can approximate this expression to find the ratio on the energy levels perturbations and Bohr levels as follows:

$$\frac{\Delta E_{n,l}}{E_n} = -\frac{\nu}{nl} + O(\nu^2). \quad (38)$$

### A. Classical energy perturbation

Now we consider the effect of a classical perturbation given by the extra potential  $\delta V(r) = -\alpha m / (2r^2)$  on the energy of Kepler's elliptical orbits  $E = -GMm / (2a)$ . The time average of the perturbing potential in the unperturbed Keplerian orbit is given by:

$$\langle \delta V \rangle = \frac{\alpha}{2T} \int_0^T \frac{dt}{r^2}. \quad (39)$$

Where  $T$  is the orbital period. This integral can be more readily calculated if we use the orbital equation and the equation of time in terms of the eccentric anomaly [15]:

$$r = a(1 - \varepsilon \cos \eta), \quad (40)$$

$$\frac{2\pi dt}{T} = (1 - \varepsilon \cos \eta) d\eta. \quad (41)$$

From Eqs. (39)-(41) we can perform the time integral over the eccentric anomaly:

$$\langle \delta V \rangle = \frac{\alpha m}{4\pi a^2} \int_0^{2\pi} \frac{d\eta}{1 - \varepsilon \cos \eta} = \frac{\alpha m}{2a^2 \sqrt{1 - \varepsilon^2}}. \quad (42)$$

Where the semi-major axis is given by:

$$a = n^2 a_0 = n^2 \frac{\hbar^2}{GMm^2}. \quad (43)$$

And we have used the expression in Eq. (5) for the Bohr's radius,  $a_0$ , of the gravitational system. From Eqs. (6), (42) and (43) we can calculate the quotient among the

perturbation and the energy of the original unperturbed orbit as follows:

$$\frac{\langle \delta V \rangle}{E} = -\frac{\alpha}{GMa\sqrt{1 - \varepsilon^2}} = -\frac{\alpha m^2}{\hbar^2 nl} = -\frac{\nu}{nl}. \quad (44)$$

Which coincides with the quantum result in Eq. (38) to first order in  $\nu$ .

## IV. CONCLUSIONS

Bohr's correspondence principle has largely influenced the development of quantum mechanics and its interpretation [17]. Even on the early quantum theory, Bohr was already interested in explaining the connection of the novel quantum conditions from a classical point of view. It was shown that the angular momentum quantization  $L = n\hbar$  was compatible with the emission and absorption of photons with a frequency given as an integer multiple of  $1/T$ ,  $T$  being the orbital period for  $n$  large. This suggested a promising avenue to recover a result of classical electrodynamics, *i. e.*, a charged orbiting body radiates with frequencies in multiples of the orbiting frequency. This frequency interpretation of the correspondence principle appeared very early in the development of the theory. In another interpretation, the intensity of the classical radiation was related to quantum emission probabilities. A third interpretation stated that every allowed transition by the quantum selection rules corresponds to one harmonic component of the classical motion [18].

In many texts of quantum physics, however, the correspondence principle takes a broader perspective as the convergence of the classical and quantum predictions in the case of large quantum states, *i. e.*, those widely spread in space and corresponding to bodies of large mass. However, scarce dedication to this issue is given in textbooks, despite its pedagogical and conceptual interest and recent experimental developments on this subject [8, 9]. The one-dimensional harmonic oscillator is usually the only system studied for large quantum numbers.

In this paper we have analyzed the most important exactly solvable problem of Schrödinger's quantum mechanics, *i. e.*, the hydrogen atom, a single particle orbiting towards a center of mass attracted by Coulomb force. In the classical limit this problem should correspond to a Solar system problem and we expect to recover the characteristic orbital parameters not present in Born's probability density for small values of the quantum numbers. We have shown how for large values of the angular quantum number,  $l$ , the angular factor in Born's probability density restricts the state to a plane in agreement with the classical property of planar motion in Kepler's problem. Moreover, for large  $n$  the radial part is confined to a very precise radius and, consequently, the quantum state has a very narrow toroidal shape in correspondence with classical circular orbits. Classical orbital velocity can also be deduced as the quotient of the quantum flux and Born's

probability density.

A similar approach to the elliptical orbits lead to the conclusion that the quantum state in the macroscopic limit is the result of a time average over all orbits sharing the same invariants: energy and angular momentum. The quantum state resembles a flattened disk with an inner hole because the classical region  $r < a(1 - \epsilon)$  becomes forbidden as the particle cannot be found at distances from the center smaller than the perihelion.

Finally, we have shown that perturbations in the eigenvalue of the energy can be computed as time orbital averages of the perturbation potential over the unperturbed orbit of the Newtonian potential. We find that the correspondence with quantum eigenstates in the Kepler's problem is achieved as an average over all possible classical orbits with the same energy and angular momentum vector. The quantum probability density coincides then with the probability to find the particle in a given position at a randomly chosen instant of time. We show that in Quantum Mechanics all possible configurations have the same status of reality and coexist to define the stationary states.

An interesting extension of this work would be the understanding of macroscopic quantum states in a classical gravitational field in General Relativity in order to disclose the correspondence principle for quantum fields in classical curved spacetime. Further work along this line is being carried out.

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