Modeling the one-dimensional oscillator with variable mass

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Abstract

We discuss the general form of Newton's second law valid for describing the dynamics of one-dimensional variable mass systems. We thus obtain the equation of motion of the one-dimensional oscillator with a variable mass, which is modeled as a quadratic function of time. The obtained equation of motion is numerically solved by means of a simple procedure. The work is addressed to physics courses at undergraduate level.

Resumen

Se discute la forma general de la segunda ley de Newton válida para describir la dinámica de los sistemas de masa variable de una dimensión. Obtenemos así la ecuación de movimiento del oscilador unidimensional con una masa variable, que se modela como una función cuadrática del tiempo. La ecuación obtenida de movimiento es numéricamente resuelta por medio de un procedimiento simple. La obra está dirigida a los cursos de física a nivel de pregrado.

Keywords: Newton’s second law, variable mass systems, oscillators.

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I. INTRODUCTION

In mechanics, variable-mass systems are systems which have mass that does not remain constant with respect to time. In such systems, Newton's second law of motion cannot directly be applied because it is valid for constant mass systems only [1, 2]. Instead, a body whose mass m varies with time can be described by rearranging Newton's second law and adding a term to account for the momentum carried by mass entering or leaving the system [1, 6].

Due to some conceptual difficulties, this topic is not commonly addressed in basic physics courses. Thus, it may be interesting to propose new approaches to the topic for students of science and engineering at undergraduate level.

In this work, we derive the correct form of Newton's second law applied to single-degree of freedom systems with a time-variable mass. Then, we describe the dynamics of the single one-dimensional oscillator with the mass modeled by a quadratic function of time. The obtained equation of motion is solved by using a suitable numerical procedure for given initial conditions.

The work is mainly addressed to undergraduate students and teachers. The study of this topic requires acquaintance with basic concepts of calculus and physics at intermediate level.

II. NEWTON'S SECOND LAW FOR VARIABLE MASS SYSTEMS

Consider a particle of mass m which is moving with velocity v at time t. Under the action of the force F between the time instants t and $t + dt$, its velocity changes from v to $v + dv$. According to Newton's second law, the change of the linear momentum, dp, is given by

$$dp = Fdt, \tag{1}$$

where $p=mv$. For constant mass, the previous equation entails

$$m\frac{dv}{dt} = F. \tag{2}$$

Equation (2) represents Newton's second law commonly presented in textbooks. This form is particularly useful in obtaining the equation of motion of a constant mass particle. Equation (1) leads to an alternative form of Newton's second law, however:

$$F = \frac{dp}{dt} = \frac{d}{dt}(\mathrm{mv}). \tag{3}$$

When applied to describe the dynamics of a constant mass particle, equations (2) and (4) provide equivalent expressions of Newton's second law. Furthermore, those equations are invariant under the Galilean transformation, defined by

$$x' = x - ut, \qquad v' = v - u, \qquad (4)$$

where u is the velocity of the primed frame of reference relative to the unprimed frame.

The description of a variable-mass system is a bit more difficult. In order to explore this point, let us apply the Galilean transformations (4) to the equation of motion (3).

The invariance of Newton's second law enforces that the equation of motion in the primed frame of reference must retain the same form of equation (3):

$$F' = \frac{d}{dt}(mv'). \qquad (5)$$

The time derivative in the right-hand side yields

$$F' = \mathrm{m}\frac{\mathrm{dv}}{\mathrm{dt}} + \frac{dm}{dt}(v - u). \qquad (6)$$

and hence

$$F' = \frac{\mathrm{d}}{\mathrm{dt}}(mv) - \frac{dm}{dt}u \neq F. \qquad (7)$$

So, the equation of motion (3) is not Galilean invariant when the particle mass is time dependent. In order to properly obtain the equation of motion, we have to apply the principle of conservation of linear momentum for the entire system, which is the basic principle behind the Newton's second law. Thus, consider a single-degree of freedom system with a time varying mass m, as illustrated in Fig. 1.

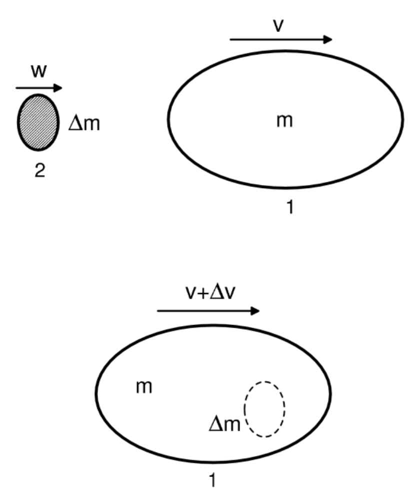

FIGURE 1. The particle of mass Δm and velocity w collides with a particle of mass m and gets stuck in it (upper part). After the process, the new particle of mass $m + \Delta m$ moves with velocity v + Δv (lower part).

The system (the body labeled 1 in the figure) moves with velocity v at the time t. The particle of mass Δm (labeled 2 in the figure) and mean velocity w collides with the system during a time interval Δt, imparting mass to the system. Assuming that the mass of the entire system is conserved during the collision, the new mass and the new velocity of the original system increase to $m + \Delta m$ and to $v + \Delta v$, respectively. The linear momentum of the system at the time t is thus given by $p(t) = mv + (\Delta m)w$, while the new linear momentum at the time $t + \Delta t$ reads $p(t+\Delta t) = (m + \Delta m)(v + \Delta v)$. Hence, the change in the total linear momentum is

$$\frac{\Delta p}{\Delta t} = m\frac{\Delta v}{\Delta t} + \frac{\Delta m}{\Delta t}\Delta v - \frac{\Delta m}{\Delta t}(w - v). \qquad (8)$$

Taking the limit $\Delta t \to 0$, $\Delta m \to 0$, $\Delta v \to 0$ in equation (8), one arrives to

$$\frac{dp}{dt} = m\frac{dv}{dt} - \frac{dm}{dt}(w - v), \qquad (9)$$

where $w - v$ is the velocity of the incoming mass with respect to the centre of mass. Combining equations (3) and (9) one obtains

$$F = m\frac{dv}{dt} - \frac{dm}{dt}(w - v), \qquad (10)$$

which in turn can be put in the form

$$m\frac{dv}{dt} = F + \frac{dm}{dt}(w - v), \qquad (11)$$

F being the net external force acting on the system. Analogously, for the case $dm/dt<0$ (system losing mass) we would obtain

$$m\frac{dv}{dt} = F - \frac{dm}{dt}(w - v), \qquad (12)$$

Equations (11) and (12) describe the motion of a time varying mass particle, and represent the proper extension of Newton's second law. The term $\frac{dm}{dt}(w - v)$ in the right-hand side of both quoted equations should be interpreted as a real force acting on the particle, apart from the external force F. For the particular case $F=0$, the equation (12) leads to the simplified equation of motion

$$m\frac{dv}{dt} = -\frac{dm}{dt}(w - v), \qquad (13)$$

Equation (13) is known as the rocket equation, which describes the motion of rockets drifting in the free space. The relative velocity w-v represents the velocity of gases escaping from the rocket, and is often called the *exhaust velocity*, and denoted by v_e [5, 7].

Also note that equation (10) may be put in the form

$$F = \frac{d}{dt}(mv) - \frac{dm}{dt}w, \qquad (14)$$

which entails that equation (14) recovers equation (3) in the particular case $w=0$.

Finally, it is easy to verify that equation (14) is invariant under Galilean transformation.

III. ENERGY BALANCE

The kinetic energy of the system at the time t is given by

$$T(t) = \frac{1}{2}mv^2 + \frac{1}{2}\Delta m w^2, \tag{15}$$

while at the time $t + \Delta t$ the kinetic energy is

$$T(t + \Delta t) = \frac{1}{2}(m + \Delta m)(v + \Delta v)^2. \tag{16}$$

Neglecting higher order terms in Δm and Δv, the change in the kinetic energy reads

$$\Delta T = mv\Delta v + \frac{1}{2}\Delta m(v^2 - w^2), \tag{17}$$

which leads to

$$\frac{dT}{dt} = m\frac{dv}{dt}v + \frac{1}{2}\frac{dm}{dt}(v^2 - w^2). \tag{18}$$

Inserting (11) into the right-hand side of equation (18), we obtain the power supplied by the force F, which reads

$$Fv = \frac{dT}{dt} + \frac{1}{2}\frac{dm}{dt}(w - v)^2. \tag{19}$$

The time derivative of the kinetic energy of the particle 1 is explicitly given by

$$\frac{dT_1}{dt} = m\frac{dv}{dt}v + \frac{1}{2}\frac{dm}{dt}v^2, \tag{20}$$

from which equation (18) can be put in the form

$$\frac{dT}{dt} = \frac{dT_1}{dt} - \frac{1}{2}\frac{dm}{dt}w^2. \tag{21}$$

Then, inserting (21) into the right-hand side of (19) one obtains

$$Fv = \frac{dT_1}{dt} - \frac{1}{2}\frac{dm}{dt}w^2 + \frac{1}{2}\frac{dm}{dt}(w - v)^2. \tag{22}$$

Notice that for the case $\frac{dm}{dt} < 0$ (system losing mass) we would obtain instead

$$Fv = \frac{dT_1}{dt} + \frac{1}{2}\frac{dm}{dt}w^2 - \frac{1}{2}\frac{dm}{dt}(w - v)^2. \tag{23}$$

In order to analyze the role of each term present in the right-hand side of equations (22) and (23), consider for the sake of illustration a one-dimensional system with only two forms of energy: the kinetic energy T and the internal energy U. (The reasoning used hereinafter in this section are not completely valid for the system described in the next sections, where other forms of energy are present, like gravitational potential energy and elastic potential energy.) So, the total energy can be written as $E=T+U$. Assuming no heat transfer between the system and the external environment it follows that

$$Fv = \frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt}. \tag{24}$$

Comparing equations (19) and (24) we get

$$\frac{dU}{dt} = \pm\frac{1}{2}\frac{dm}{dt}(w - v)^2, \tag{25}$$

where the sign + (−) stands for $\mathrm{dm/dt} > 0$ (< 0).

Equations (24) and (25) tell that the work done by the external force F is not entirely converted into kinetic energy.

Regardless whether the system is losing or acquiring mass, part of that work is transformed into internal energy. In short: a particle of variable mass behaves like a dissipative system.

If we ignore external forces when two masses hit and stick together, it follows from (24) that

$$\frac{dT}{dt} = -\frac{dU}{dt}. \tag{26}$$

Equation (26) refers to totally inelastic collisions, which entails that the change in kinetic energy during this class of collisions is negative. This explains what happens to the kinetic energy: it mostly goes into internal energy and heat, and some of it goes into sound waves, etc.

IV. MODELING THE VARIABLE MASS OSCILLATOR

In order to model the variable mass oscillator, consider a leaking bucket of water which is attached to a spring, as illustrated in Fig. 2. The water exits out the bucket through a small hole at the bottom. Assume that the mass loss of water and the motion of the oscillator are along a line (the z-axis).

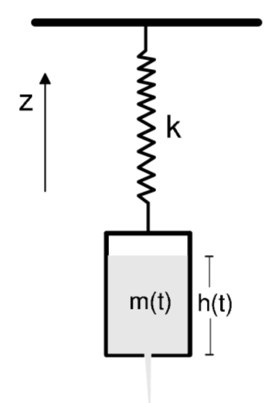

FIGURE 2. Oscillator with a variable mass. A bucket of water which is attached to a spring. The water flows out through a small hole in the bottom of the bucket.

In this situation, and ignoring friction, the system is subjected to the action of three different forces, namely, the elastic force exerted by the spring, the weight of the oscillator, and the force exerted by the leaking water. In accordance with equation (12), one obtains [6]:

$$m\frac{d^2z}{dt^2} = -\frac{dm}{dt}q - kz - mg, \quad (27)$$

where $q = w - v$, *z(t)* is the displacement of the centre of mass measured from the initial equilibrium position; *w* is the mean velocity at which the water leaves the system; $v = dz/dt$ is the velocity of the oscillator; k is the stiffness coefficient of the linear restoring force; and g is the acceleration of gravity.

Now, imposing the conditions $w = v = 0$, $\frac{dm}{dt} = 0$, and $\frac{d^2z}{dt^2} = 0$ at the time $\mathrm{t} = 0$, provide the equilibrium position

$$z_0 = -\frac{m(0)g}{k}, \quad (28)$$

where $m(0)$ is the initial mass of the oscillator. If m is constant, the system would oscillate around the equilibrium position z_0. So, by means of the transformation

$$z \rightarrow z + z_0. \quad (30)$$

The equation (27) turns into

$$m\frac{d^2z}{dt^2} = -\frac{dm}{dt}q - kz + (m(0) - m)g. \quad (31)$$

So, the "instantaneous" equilibrium position at every time *t* is computed by

$$z_0(t) = \frac{m(0) - m(t)}{k}g, \quad (32)$$

which entails that as the water leaves the leaking bucket, the equilibrium position undergoes a continuous upward movement.

The mass of water within the bucket has a quadratic dependence on the time (see appendix for details of calculation), which is given by

$$m_w(t) = m_w(0)\left(1 - ft\sqrt{\frac{g}{2h_0}}\right)^2, \quad (33)$$

where $m_w(0)$ is the initial mass of water, $f = \frac{a}{A}$, is the ratio between the cross-sectional areas, and h_0 is the initial height of the column of water. The mass of the oscillator is given by the summation of the mass of the bucket m_b, and the time-varying mass of water $m_w(t)$.

Assuming the leaking of water occurs at a very low rate, one can neglect the effect of the first term on the right side of equation (31) on the dynamics of the oscillator. The equation of motion thus reads

$$(m_b + m_w)\frac{d^2z}{dt^2} = -kz + (m_w(0) - m_w)g. \quad (34)$$

V. NUMERICAL SOLUTION

According to equation (33), the bucket of water is completely empty after the elapsed time given by

$$\tau = \frac{1}{f}\sqrt{\frac{2h_0}{g}}. \quad (35)$$

For the time interval $0 \leqq t \leqq \tau$ the equation of motion (34) can be put in the form

$$a = \frac{d^2z}{dt^2} = -\frac{kz - (m_w(0) - m_w)g}{m_b + m_w}, \quad (36)$$

After the elapsed time τ, the oscillations are governed by the equation of motion

$$a = \frac{d^2z}{dt^2} = -\frac{kz - m_w(0)g}{m_b}, \quad t \geqq \tau. \quad (37)$$

Henceforth we discuss approximate solutions of equations (36) and (37). With this aim, we use a simple numerical method which can be implemented, for example, in electronic calculators or even by using the Excel spreadsheet.

Thus, consider a generic function of time $y(t)$. We can assign to the time derivative of $y(t)$ the approximate expression

$$\frac{dy}{dt} \approx \frac{y(t+h) - y(t)}{h}, \quad (38)$$

which can be a good approximation if the time step *h* is small enough. This allows to compute the approximate value of *y* at the time $t + h$:

$$y(t+h) \approx y(t) + h\frac{dy}{dt}, \quad (39)$$

provided the time derivative of *y* at the time *t*.

Thus, consider the first order differential equation

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0. \quad (40)$$

Euler method consists of using (39) as the approximate solution of the differential equation (40). So, assign a value for the time step *h*, and set $t_0 + h$. Now, the first step from t_0 to $t_1 = t_0 + h$ yields the new value of *y*:

$$y_1 = y(t_0 + h) = y_0 + hf(t_0, y_0). \quad (41)$$

After *n* successive steps, we get

$$y_{n+1} = y_n + hf(t_n, y_n), \quad (42)$$

where $y_n = y(t_n)$.

Further improvements of the Euler method allows approximate solutions of higher accuracy. One of these improvements is the midpoint method, which consists of taking function evaluations at the time step $t_n + \frac{h}{2}$, and then applying the Euler method using these evaluations. So, following this method we have first to compute

$$y_{n+1/2} = y_n + \frac{h}{2} f(t_n, y_n), \quad (43)$$

and then obtain the value of the function at the time $t_n + h$:

$$y_{n+1} = y_n + hf\left(t_{n+1/2}, y_{n+1/2}\right), \quad (44)$$

where by definition $t_{n+1/2} = t_n + h$. Note that in equation (44) the derivative function $f(t, y)$ is evaluated at the midpoint time $t_n + \frac{h}{2}$, and not at the time t_n.

In order to numerically solve equation (36) valid for the time interval $0 \leqq t \leqq \tau$, we first compute $z(t)$ and $v(t)$ at the time step $t_n + \frac{h}{2}$:

$$z_{n+1/2} = z_n + \frac{h}{2} v_n, \quad (45)$$

and

$$v_{n+1/2} = v_n + \frac{h}{2} a_n, \quad (46)$$

where

$$a_n = -\frac{kz_n - (m_w(0) - m_w(t_n))g}{m_b - m_w(t_n)}, \quad (47)$$

with $z_n = z(t_n)$ and $v_n = v(t_n)$. The next step provides:

$$z_{n+1} = z_n + hv_{n+1/2}, \quad (48)$$

and

$$v_{n+1} = v_n + ha_{n+1/2}, \quad (49)$$

where now

$$a_{n+1/2} = -\frac{kz_{n+1/2} - (m_w(0) - m_w(t_{n+1/2}))g}{m_b - m_w(t_{n+1/2})}. \quad (50)$$

In order to compute the dynamical evolution of the oscillator for $t \geqq \tau$, the equation (37) must be replaced by

$$a_n = -\frac{kz_n - m_w(0)g}{m_b}, \quad (51)$$

at the time t_n.

VI. RESULTS

In this section we present results obtained for the case of a bucket of mass $m_b = 1.0$ kg, filled with the initial mass of water of $m_w(0) = 10$ kg with a column of initial height $h_0 = 0.5$ m. The bucket of water is attached to the spring of stiffness coefficient $k = 100$ N/m. The mass of the oscillator is given at every time by $m(t) = m_b + m_w(t)$, $m_w(t)$ being the time varying mass of water.

The algorithm for solving the equations (36), (37) comprises the following steps. First, assign initial values to all variables: the elapsed time $t = 0$; the initial position $z = z(0)$; the initial velocity v(0); the initial mass of water $m_w(0)$; the initial height of the water column h_0; the value of the ratio between the cross-sectional areas $f = a/A$. Assign values to constants g, k, and the mass of the bucket m_b. Then, compute the position $z_{n+1/2}$ at the time step $t_n + h/2$, given by equation (45); compute the acceleration step a_n at the time step t_n, given by equation (47) for $0 \leqq t \leqq \tau$, and by equation (50) for $t \geqq \tau$; then, compute the velocity $v_{n+1/2}$ at the time $t_n + h/2$, given by equation (46). Compute the acceleration $a_{n+1/2}$ given by equation (50), or by (51) if $t \geqq \tau$. Now, compute the new values of v and z by using equations (49) and (48). The time t is incremented by h at each step. The new values of the mass of water within the bucket $m_w(t)$ is directly obtained from equation (33) as the function of time.

We start with the initial condition $z(0) = 0$ and $v(0) = 0$. Thus, the change of the dynamical state of the system is initially caused by the change in mass of the oscillator with time. We also compute the elastic potential energy U_k, and the gravitational potential energy, W, which are given respectively by

$$U_k = \frac{1}{2} k \left(z - \frac{m_0 g}{k}\right)^2,$$

and

$$W = \left(m_b + m_w(t)\right)g.$$

The mechanical energy of the system is given by the summation of the elastic potential energy, the gravitational potential energy and the kinetic energy, namely $E = T + W + U_k$.

In the carried out numerical simulations we adopt the value $h = 0.01$ s for the time step. With this step we need a few thousand steps to perform the simulation. Assigning for the ratio $f = a/A$ the value $f = 0.01$, for example, the water takes tens of seconds to exit the leaking bucket. So, this does not demand a huge computational time, providing however a very accurate numerical result.

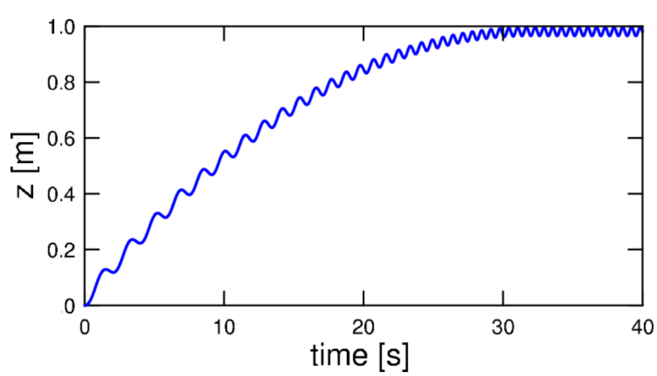

FIGURE 3. Position as a function of time for the value *f=0.01*. The used values of the other parameters are $= 9.8\ m.s^{-2}$, $k = 100\ N/m$, $m_b = 1.0\ kg$ (mass of the bucket), $m_w(0) = 10\ kg$ (initial mass of water), and $h(0) = 0.5\ m$ (initial height of the water column).

Figure 3 depicts the behavior of the position of the oscillator as a function of time for the adopted values of the model parameters outlined in the caption of the figure. Notice that the "instantaneous" equilibrium position of the oscillator moves upward while the water within the bucket flows out.

The oscillations are obviously caused by the action of the restoring force, as the mass of the oscillator decreases. In special, one notices that the amplitude of the oscillations decreases, while the frequency increases as the mass of the oscillator decreases. The final equilibrium position, around which the system oscillates for $t > \tau$, can be computed by using equation (28), which in the present case has the value $0.98\ m$.

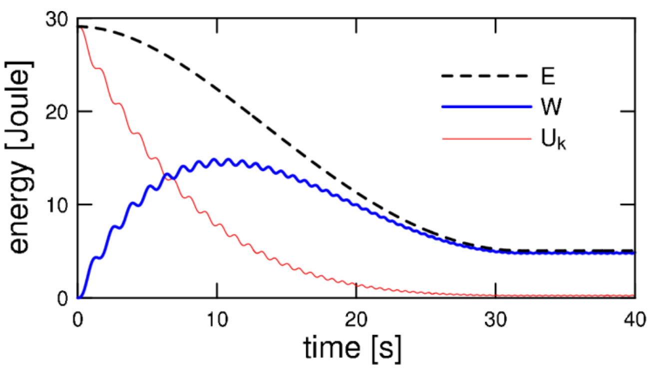

FIGURE 4. Energy as a function of time for $f = 0.01$. The values of the other model parameters are the same used in the previous figure.

Figure 4 shows the behavior of the energy of the oscillator as a function of time for the same set of values of the parameters used in Figure 3. As discussed in Section III we can see that the total energy of the oscillator is dissipated due to the mass loss of the system.

VI. CONCLUSIONS

In this work, we discuss the appropriate form of Newton's second law applied to single-degree of freedom systems with a time variable mass. We present a set of equations which are used to model the dynamics of a one-dimensional oscillator with a time-varying mass. The dependence of the mass on the time is taken into account, by means of a simple modeling (the leaking bucket of water) where the mass of the oscillator has a quadratic dependence on time.

The resulting equation of motion is numerically solved in terms of the improved Euler method, and some results for chosen values of the model parameters have been presented and discussed in the text.

According to the results obtained by the numerical simulations, the system shows a typical oscillatory behavior with "amplitude" and "frequency" which vary as the water leaves the bucket. At the end, there remains only the bucket that oscillates like a one-dimensional harmonic oscillator with constant amplitude and frequency.

This study, despite its simplicity, is intended to be used as a useful approach for students get acquainted with the physics of systems with time-varying mass at the undergraduate level. The conclusions must notice the new and remarkable contributions of the paper. Also the suggestions and shortcomings of the manuscript must be pointed out.

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APPENDIX

Let us thus consider a cylindrical bucket of water with cross-sectional area A with a column of water of height h. At the bottom of the bucket there is a small hole with cross-sectional area a, with $a \ll A$, through which the liquid flows out under the action of the gravity force.

As depicted in Figure 2, we place the point of reference 2 at the free liquid surface, and the reference 1 at the bottom of the bucket. Neglecting losses, which is reasonable if the hole is tiny and the storage bucket is large and wide, we can apply the Bernoulli equation:

$$p_2 + \frac{1}{2}\rho Q^2 + \rho g(z_1 + h) = p_1 + \frac{1}{2}\rho q^2 + \rho g z_1, \quad \text{(A1)}$$

where p_1 and p_2 are the pressure at the bottom of the bucket and at the free liquid surface, respectively. The upper part of the bucket is open to the atmosphere, and the water leaks the bucket freely through the hole in the bucket bottom. So, we have $p_1 = p_2 = p_0$, where p_0 is the local atmospheric pressure. Q is the velocity at the free liquid surface and q is the exit velocity of the water; h is the height of the free liquid surface relative to the bottom; ρ is the density of the liquid; and z_1 is the position of the bottom of the bucket in along the *z*-axis.

Because of the large cross-sectional area A in comparison with the hole, the velocity Q can be set equal to zero. Thus, we can make these substitutions into the Bernoulli equation to obtain

$$q = \sqrt{2gh}\,. \tag{A2}$$

Notice that equation (A2) is valid even when the surface level is decreasing due to water leakage, provided that the time rate of change of h and Q is sufficiently small. The rate at which the height h decreases with time can be used to calculate the rate of loss of mass, applying the mass balance on the content of the bucket. In fact, from the equation of continuity the rate of loss of mass is related to the mass flow trough the equation

$$\frac{dm}{dt} = -\rho q a. \tag{A3}$$

On the other hand, the mass of water stored in the bucket at time *t* is given by

$$m(t) = \rho A h(t). \tag{A4}$$

where $h(t)$ is the height of the water column within the bucket.
Inserting (A2) into (A3) yields

$$\frac{dm}{dt} = -\rho a\sqrt{2gh(t).} \tag{A5}$$

From (A4), one can put (A5) in the form

$$\frac{dm}{dt} = -f\sqrt{\rho A}\sqrt{2gm}, \tag{A6}$$

where $= \frac{a}{A}$. Thus we find

$$\frac{dm}{m^{1/2}} = -f\sqrt{\rho A}\sqrt{2g}\,dt. \tag{A7}$$

Integrating the equation (A7), we obtain the mass of water within the bucket as a function of time, given by equation (33).