

Flying off a frictionless curved ramp



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Abstract

A general equation is derived for the point along a descending curved track that a particle (such as a marble or model car) loses contact with the surface, in the absence of dissipation of mechanical energy. The object is assumed to start from rest at a point on the curve of zero slope. The launch speed and angle are calculated for the examples of a circular and a log-secant curve. The equation shows that a particle can never launch off a parabolic track, because it cannot attain the speed of a freefalling object skimming along the surface of such a track.

Keywords: Roller coaster track, Centripetal acceleration, Conservation of mechanical energy.

Resumen

Una ecuación general se deriva para un punto a lo largo de una trayectoria curvada descendente, de una partícula (tal como una bolita o un carrito) que pierde el contacto con la superficie, en ausencia de disipación de la energía mecánica. Se supone que el objeto empieza desde el reposo hasta un punto en que la pendiente de la curva es cero. La velocidad de lanzamiento y el ángulo se calculan para los ejemplos de una curva circular y una curva log-secante. La ecuación muestra que una partícula nunca puede lanzarse fuera de una trayectoria parabólica, porque no puede alcanzar la velocidad de un objeto en caída libre que coincide superficialmente con una trayectoria de este tipo.

Palabras clave: Trayectoria de la montaña rusa, Aceleración centrípeta, Conservación de la energía mecánica.

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I. INTRODUCTION

A standard textbook problem [1] consists in finding the point at which a particle slides off a hemispherical surface (such as a giant snowball) starting from rest at the top, ignoring friction and air drag. Similarly, for any given curved ramp, what are the coordinates (if any) at which contact with the surface is lost? What then is the particle's launch velocity (*i.e.*, speed and direction)?

For simplicity, assume that the particle always starts from rest at a point on the curve that has zero slope, so that a slight nudge is needed to get it sliding down the ramp.

Define that starting point to be the origin (0,0). Restrict attention to plane curves, with $+x$ in the direction of initial motion and $+y$ in the vertically downward direction of the local gravitational field having magnitude g .

The particle loses contact with the track when it has speed v_0 and when the slope of the ramp makes angle θ_0 measured clockwise from the $+x$ axis in Fig. 1 (*i.e.*, toward the direction of the $+y$ axis, in accord with the usual convention for angle in plane polar coordinates).

II. EQUATION FOR THE POINT AT WHICH THE PARTICLE LOSES CONTACT WITH THE SURFACE

A free-body diagram is sketched in Fig. 1 for a particle of mass m that is sliding along a ramp in the absence of friction and air resistance. There are two forces acting on the particle: the normal force N that the ramp exerts perpendicularly to itself and the downward gravitational force mg .

The acceleration of the particle has two orthogonal components: the centripetal acceleration $a_c = v^2 / r$ perpendicular to the track where v is the speed of the particle at the instant indicated and r is the radius of curvature of the track at that point, and the tangential acceleration $a_t = dv / dt$ parallel to the track that measures the increase in speed of the particle with the passing of time t .

The particle loses contact with the surface when the normal force in Fig. 1 is zero. In that case, the component of Newton's second law perpendicular to the track becomes:

$$F_c = ma_c \Rightarrow mg \cos \theta = m \frac{v^2}{r}, \quad (1)$$

where θ is the angle the track makes relative to the horizontal, assumed to be restricted to the range $-\pi/2 < \theta < \pi/2$ (thereby excluding internal loop-the-loops). Suppose the normal force first becomes zero when the particle is a vertical distance y below the origin. From conservation of mechanical energy, the speed v of the particle at that instant is:

$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}. \quad (2)$$

The radius of curvature [2] can be mathematically calculated for a plane curve as:

$$r = \frac{(1 + y'^2)^{3/2}}{y''}, \quad (3)$$

where the primes indicate derivatives with respect to x . (For example, a straight track has $y'' = 0$ so that $r \rightarrow \infty$ and hence $a_c = 0$. A particle thus never loses contact with a straight section of track inclined at less than the vertical.) Finally, because the slope of a curve can be written as: $y' = \tan \theta$, it follows that:

$$\cos \theta = \frac{1}{\sqrt{1 + \tan^2 \theta}} = \frac{1}{\sqrt{1 + y'^2}}. \quad (4)$$

(The positive square root is appropriate for the range of angles assumed above.) Substituting Eqs. (2) through (4) into (1), one finds that the particle loses contact with the track at a point that satisfies the equation:

$$y = \frac{1 + y'^2}{2y''}. \quad (5)$$

A similar equation has been obtained to describe the motion of a bead sliding frictionlessly along a cycloidal wire [3].

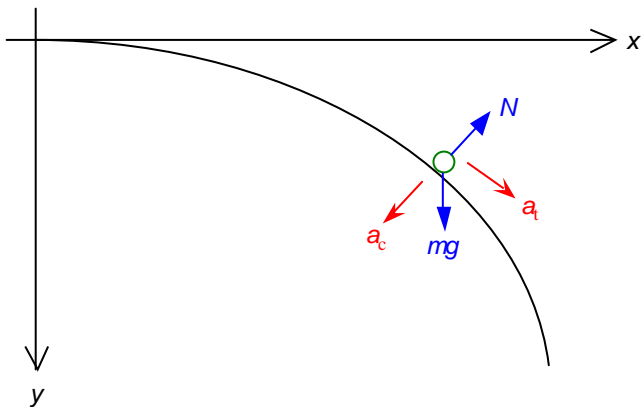


FIGURE 1. A particle sliding frictionlessly along a plane curve $y(x)$. The particle started with negligible initial speed at the origin (where the slope of the curve is zero).

III. APPLICATION TO THREE PARTICULAR TRACK SHAPES

As a check on Eq. (5), first apply it to a semicircular track of radius R whose top point is at the origin,

$$y(x) = R - \sqrt{R^2 - x^2}. \quad (6)$$

(Substituting this expression into Eq. (3) gives $r = R$ as expected). Inserting Eq. (6) into (5) and simplifying, one correctly finds the standard result [1] that the particle loses contact with the track at a value of y given by $y_0 = R/3$ (i.e., at a height above the equator equal to two-thirds of the radius).

The corresponding value of x at that point is $x_0 = \sqrt{5}R/3$ and the launch angle is:

$$\tan \theta = \frac{dy}{dx} \Rightarrow \theta_0 = \tan^{-1} \sqrt{1.25} \approx 48^\circ, \quad (7)$$

which is also the polar spherical angle (measured clockwise from the top of the track) at which the particle flies off. The launch speed is:

$$v_0 = \sqrt{\frac{2gR}{3}}, \quad (8)$$

according to Equation (2).

As a second application, consider a parabolic track whose vertex is at the origin $y = kx^2$, with $k > 0$ so that the particle can slide down it. Substituting that function into Eq. (5), one finds that there is no solution for finite k . In other words, a particle can never lose contact with an inverted parabolic track if it starts at the vertex at rest.

The reason for this behavior is that a particle in freefall would skim along the surface of the curve if it were projected horizontally from the origin with an initial speed of $\sqrt{g/2k}$.

Since our particle instead starts from rest, its speed at any given point along the curve will always be lower than that of the freefalling particle at the same point, and hence our particle will never be able to make a transition to freefall.

The preceding two examples began with tracks of definite shapes. Another approach is to look for general solutions of Equation (5). However, that equation only specifies a relationship between the three values of y , y' , and y'' at the launch point. It does not constrain their values at other points along the curve. To make progress, one or more additional conditions must be imposed. For instance, we can fix the value of y in Eq. (5), say to $y_0 > 0$, and then solve the differential equation. This constraint requires a specific proportionality to hold between the values of y' and y'' at all points along the curve, namely:

$$1 + y'^2 = 2y_0 \frac{dy'}{dx}. \quad (9)$$

Separating and integrating, one obtains:

$$x = 2y_0 \tan^{-1}(y') \Rightarrow \frac{dy}{dx} = \tan\left(\frac{x}{2y_0}\right), \quad (10)$$

where the constant of integration is zero to ensure $y'(0) = 0$. Separate and integrate again to get:

$$y(x) = 2y_0 \ln \left[\sec \left(\frac{x}{2y_0} \right) \right], \quad (11)$$

after requiring the curve to pass through the origin. Defining the numerical value:

$$A \equiv 2 \cos^{-1} \left(e^{-1/2} \right) \approx 1.838, \quad (12)$$

the particle loses contact with this log-secant curve at coordinates (Ay_0, y_0) . The launch velocity is $v_0 = \sqrt{2gy_0}$ at an angle of:

$$\tan \theta = \frac{dy}{dx} \Rightarrow \theta_0 = \cos^{-1} \left(e^{-1/2} \right) \approx 53^\circ. \quad (13)$$

One may wonder whether a similar method could be used to find track shapes for the following constraints:

- (i) Fix a positive value of y' in Eq. (5);
- (ii) Fix a positive value of y'' in Eq. (5); or
- (iii) Require Eq. (5) to hold at all values along the curve, not just at the launch point.

The reader is invited to verify that in all three of these cases, the solution to the resulting differential equation (obtained using the identity $y'' = y' dy' / dy$ when helpful) cannot satisfy the two boundary conditions $y(0) = 0$ and $y'(0) = 0$.

It appears to be serendipitous that we were able to find a satisfactory solution when we fixed a value of y in Eq. (5).

IV. CLOSING COMMENTS

Two examples of launch curves that have zero slope at the origin are a semicircular track and a log-secant ramp. For both of them, the launch speed can be varied by adjusting the overall scale size of the track. However, the launch angle is fixed regardless of this chosen scaling see Eqs. (7) and (13).

A challenging topic for further investigation is to find a general equation for a smoothly varying function (that starts with zero slope at the origin) that launches a particle with any specific velocity (not just speed) of interest. Is the solution to that problem unique? If not, what additional constraints have to be placed on the track shape (or on the conditions of launch) so that one specific functional curve uniquely satisfies the constraints?

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