# How far and fast does heat propagate?

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**Abstract**

Unlike wave motion, where the propagation speed is well defined, in diffusive processes this quantity is not clearly established. In this way, any physicists can rapidly estimate the distance travelled by the light in vacuum or by the sound in air in a given time interval. However, few of them would be able to answer to the question of how far heat propagates (by conduction) inside a material in a given time. In this work, we use the concept of *thermal diffusion length* and we calculate it analytically for three situations of common life: when the sample surface of a material is put in contact with a thermal reservoir at a fixed temperature, when the surface is illuminated by a brief flash lighting and when the surface is illuminated by a continuous light beam. An easy to remember formula allows us to estimate the distance travelled by heat inside a material, which depends on its thermal diffusivity.

**Keywords:** Heat conduction, Thermal diffusivity, Heat diffusion equation.

**Resumen**

A diferencia del movimiento ondulatorio, donde la velocidad de propagación está bien definida, en los procesos de difusión esta cantidad no está claramente establecida. De esta manera, cualquier físico puede estimar rápidamente la distancia recorrida por la luz en el vacío o por el sonido en el aire en un intervalo de tiempo determinado. Sin embargo, pocos de ellos podrían responder a la pregunta de hasta qué punto se propaga el calor (por conducción) dentro de un material en un momento dado. En este trabajo, utilizamos el concepto de longitud de difusión térmica y lo calculamos analíticamente para tres situaciones de la vida común: cuando la superficie de la muestra de un material se pone en contacto con un depósito térmico a una temperatura fija, cuando la superficie se ilumina por un breve destello de luz y cuando la superficie está iluminada por un haz de luz continuo. Una fórmula fácil de recordar nos permite estimar la distancia recorrida por el calor dentro de un material, que depende de su difusividad térmica.

**Palabras clave:** Conducción de calor, Difusividad térmica, Ecuación de difusión de calor.

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## I. INTRODUCTION

The question asked in the title of this article refers to heat propagation by conduction. Actually, heat propagates at the speed of light by radiation (electromagnetic waves) and in the case of convection it greatly depends on the thermal gradient and on the geometry and orientation of the sample [1]. The mechanism responsible for heat conduction in gases and liquids is the collisions between fast molecules and slow molecules that result in an energy transfer (heat) from hot regions to colder ones. In solids, thermal conduction involves the transport of the vibrational energy of the molecules across the crystal lattice of the solid (phonons). In the case of electrically conducting solids (metals and alloys) there is an additional mechanism due to free electrons, which behave in a similar manner as the molecules in a gas. In all cases, heat conduction is a diffusive and irreversible process of statistical nature.

All physicists have an accurate idea on the propagation speed of waves, e.g. $3\times10^8$ m/s for electromagnetic waves in vacuum, 340 m/s for sound in air at room temperature and pressure, several km/s for acoustic waves in solid materials, etc. In this way, it is easy to calculate that it takes for the light a little more than eight minutes to get from the Sun to the Earth or that a storm is approximately two kilometres away from us if the time delay between lightning and thunder is six seconds. However, in diffusive processes, as is the case of heat conduction, the concept of propagation velocity is more elusive. In fact, few scientists would be able to answer apparently simple questions as: (1) how long does it take for the heat to reach the back of a wall 40 cm thick after the sun starts to light up its front face? or (2) what temperature will reach the back of that wall after being illuminated by the sun for an hour? Answering these and similar questions requires solving the heat diffusion equation, which for homogeneous and isotropic samples writes [2]

$$\nabla^2 T - \frac{1}{D}\frac{\partial T}{\partial t} = 0\,, \qquad (1)$$

where $T$ is temperature, $D = K/(\rho c)$ is the thermal diffusivity, $K$ is the thermal conductivity $\rho$ is the density and $c$ is the specific heat. It has already been pointed out that this equation predicts an infinite speed for heat propagation [3]: a sudden temperature change at some point inside a material will be felt instantaneously at each point of the sample, although with exponentially small amplitudes at distant points. Therefore, answering question (1) is tricky, since, according to Eq. (1) the right answer is zero, which is unphysical and anyway, the temperature rise would be negligible. On the other hand, the answer to question (2) depends of the solar intensity, which depends on the latitude and the hour of the day, and on the reflectivity of the wall. It is clear that there are no simple answers to these deceptively simple questions.

The aim of this paper is to help undergraduate students to answer the following question: how long does it take for the heat (by conduction) to reach a given distance inside a material? To do this, we use the concept of thermal diffusion length [4], which has been defined as the distance at which the temperature is reduced by a factor e with respect to the surface. This definition, which is similar to the exponential law of absorption of electromagnetic waves, has the advantage of being independent of the experimental conditions: power (energy) of the source, optical reflectivity, etc.

We will find analytical expressions, using undergraduate physics and mathematics, for the temperature evolution of a material whose surface is stimulated in three ways: (a) The sample surface is put in contact with a thermal reservoir at a fixed temperature, (b) the sample surface is illuminated by a brief flash lighting and (c) the sample surface is illuminated by a continuous light beam. These cases are realistic realizations of practical experiments in heat conduction. As the thermal diffusion length is similar for the three cases it allows to estimate the depth penetration in different kinds of materials.

This article is intended to serve the pedagogical purpose of using a puzzle question to attract the attention of physics students on heat propagation. It is addressed to undergraduate students in physics who have already passed through first courses of classical thermodynamics and who are familiar with the use of the Laplace transform to solve partial differential equations.

## II. THEORY

Let us consider an opaque and semi-infinite sample whose free surface is the plane $z$ = 0, as shown in Fig. 1. The sample is at room temperature and at $t$ = 0 the sample surface is heated uniformly. We consider the three kinds of surface stimulation mentioned in the introduction. To obtain the time evolution of the temperature we have to solve the one dimensional diffusion equation

$$\frac{\partial^2 T}{\partial z^2} - \frac{1}{D}\frac{\partial T}{\partial t} = 0\,. \qquad (2)$$

Along this article, $T$ represents the temperature rise above the ambient. In transient problems it is useful to work in the Laplace space [5]. In this way, the Laplace transform of the diffusion equation is

$$\frac{d^2\bar{T}}{dz^2} - q^2\bar{T} = 0\,, \qquad (3)$$

where $q^2$ = $s/D$ and $\bar{T}$ is the Laplace transform of the temperature. Equation (3) is a very well-known differential equation in physics, whose solution for a semi-infinite sample writes

$$\bar{T}(z,s) = Ae^{-qz}\,, \qquad (4)$$

where $A$ is a constant to be obtained from the boundary conditions. In the following we will calculate the constant $A$ for the three cases under study.

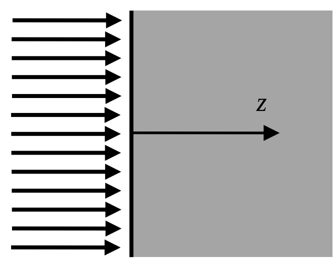



**FIGURE 1.** Diagram of an opaque and semi-infinite sample whose surface ($z$ = 0) is uniformly heated.

### A. Fixed surface temperature $T_o$

In this case, the sample surface is put in contact with a thermal reservoir at temperature $T = T_o$ above the ambient. The surface temperature is constant $T(z=0,t) = T_o$ for $t \geq 0$, whose Laplace transform is $\bar{T}(z=0,t) = T_o/s$. By substituting equation (4) into this last expression, constant $A$ is obtained and therefore the Laplace transform of the temperature at any point of the solid is obtained

$$\bar{T}(\mathrm{z},s) = \frac{T_o}{s}e^{-qz}\,, \qquad (5)$$

whose inverse Laplace transform gives the time evolution of the solid temperature

$$T(z,t) = T_o \operatorname{erfc}\left(\frac{z}{\sqrt{4Dt}}\right), \qquad (6)$$

where erfc is the complementary error function.

It is worth mentioning that Eq. (6) also governs the time evolution of the semi-infinite sample of Fig. 1 when it is put in perfect thermal contact with another semi-infinite sample ($z < 0$) made of the same material an initially at $2T_o$ above the ambient (see page 61 in Ref. [2]).

### B. Flash illumination

In this case, the surface is illuminated by a brief flash pulse of negligible duration at $t = 0$ and energy density $Q_o$ (J/m$^2$). If we neglect heat losses by convection and radiation at the sample surface the heat flux satisfies

$$\Phi(z=0) = -K \left.\frac{dT}{dz}\right|_{z=0} = Q_o \delta(t), \qquad (7)$$

whose Laplace transform is

$$\bar{\Phi}(z=0) = -K \left.\frac{d\bar{T}}{dz}\right|_{z=0} = Q_o, \qquad (8)$$

where $\delta(t)$ is the Dirac delta function. By substituting equation (4) into equation (8), the Laplace transform of the solid temperature is obtained

$$\bar{T}(\mathrm{z}, s) = \frac{Q_o}{Kq} e^{-qz}, \qquad (9)$$

whose inverse Laplace transform has an analytical expression

$$T(z,t) = \frac{Q_o}{\varepsilon\sqrt{\pi t}} e^{-\frac{z^2}{4Dt}}. \qquad (10)$$

Here $\varepsilon = \sqrt{\rho c K} = K/\sqrt{D}$ is the thermal effusivity, the quantity that measures the ability of the material to exchange heat with the environment [6,7].

### C. Continuous illumination

Now the sample surface is illuminated by a continuous light beam of intensity $I_o$ (W/m$^2$). As before, we neglect heat losses by convection and radiation at the sample surface. Accordingly, the heat flux satisfies

$$\Phi(z=0) = -K \left.\frac{dT}{dz}\right|_{z=0} = I_o, \qquad (11)$$

whose Laplace transform is

$$\bar{\Phi}(z=0) = -K \left.\frac{d\bar{T}}{dz}\right|_{z=0} = \frac{I_o}{s}, \qquad (12)$$

By substituting equation (4) into equation (12), the Laplace transform of the solid temperature is obtained

$$\bar{T}(\mathrm{z}, s) = \frac{I_o}{Kqs} e^{-qz}, \qquad (13)$$

whose inverse Laplace transform writes

$$T(z,t) = \frac{I_o}{\varepsilon}\sqrt{4t}\, \operatorname{ierfc}\left(\frac{z}{\sqrt{4Dt}}\right), \qquad (14)$$

where $\operatorname{ierfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}} - x \operatorname{erfc}(x)$, see page 484 in Ref. [2].

## III. DISCUSSION

In the following, we will show calculations of the time evolution of the temperature depth profile in the three cases analysed in the previous section. Calculations are performed for AISI-304 stainless steel whose thermal properties are: $K$ = 15 Wm$^{-1}$K$^{-1}$, $D$ = 4 mm$^2$/s [1] and therefore $\varepsilon = K/\sqrt{D}$ = 7500 Js$^{-0.5}$m$^{-2}$K$^{-1}$.

Fig. 2a shows the depth dependence of the temperature of a semi-infinite block of stainless steel whose free surface is put in contact at $t = 0$ with a thermal reservoir at 10 K above the ambient. Several times are considered. It can be observed that the surface ($z = 0$) suddenly reaches the reservoir temperature and that as time goes by heat reaches deeper regions of the steel block. From Eq. (6) and according to the definition of thermal diffusion length given in the introduction, as the distance at which the temperature is reduced by a factor $e$ with respect to the surface, we obtain

$$\frac{T(\mu,t)}{T(0,t)} = \frac{1}{e} = \operatorname{erfc}\left(\frac{\mu}{\sqrt{4Dt}}\right), \qquad (15)$$

since erfc(0) = 1. Solving Eq. (15) the thermal diffusion length is obtained: $\mu \approx \sqrt{1.62Dt}$.

Fig. 2b shows the depth dependence of the temperature of the same semi-infinite block of stainless steel as in Fig. 2a, whose free surface is illuminated by a brief flash lighting of 25 kJ/m$^2$. Five times after the laser lighting are plotted. It can be observed that at short times the surface temperature is very high but all the energy is concentrated close to the surface. At longer times, the surface temperature monotonically decreases while heat penetrates into deeper regions. From Eq. (10) together with the definition of thermal diffusion length, we obtain

$$\frac{T(\mu,t)}{T(0,t)} = \frac{1}{e} = \mathrm{e}^{-\frac{\mu^2}{4Dt}}, \tag{16}$$

indicating that the thermal diffusion length in this problem is: $\mu = \sqrt{4Dt}$ .

Finally, Fig. 2c shows the depth dependence of the temperature of the semi-infinite block of stainless steel we are dealing with, whose free surface is illuminated by a continuous light beam of intensity 1 kW/m$^2$. Five times after the laser lighting are plotted. In this case, both the surface temperature and the penetration depth are increasing with time. From Eq. (14) together with the definition of thermal diffusion length, we obtain

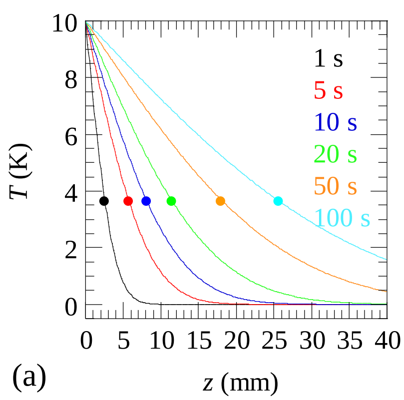
$$\frac{T(\mu,t)}{T(0,t)} = \frac{1}{e} = \frac{\mathrm{ierfc}\left(\frac{\mu}{\sqrt{4Dt}}\right)}{\pi^{-0.5}}, \tag{17}$$

where ierfc(0) = $\pi^{-0.5}$. Solving this equation, the thermal diffusion length in this problem is obtained: $\mu \approx \sqrt{0.936Dt}$ .

In Fig. 2 dots represent the thermal diffusion length at each time.

As can be seen, the thermal diffusion length is not a universal value, but it depends on the experimental conditions. Anyway, for the three cases that we have studied in this article the difference is not very high. Accordingly, we propose an intermediate and easy to remember value to estimate the penetration depth of heat in transient problems:

$$\mu_{effective} \approx \sqrt{2Dt}\,. \tag{18}$$



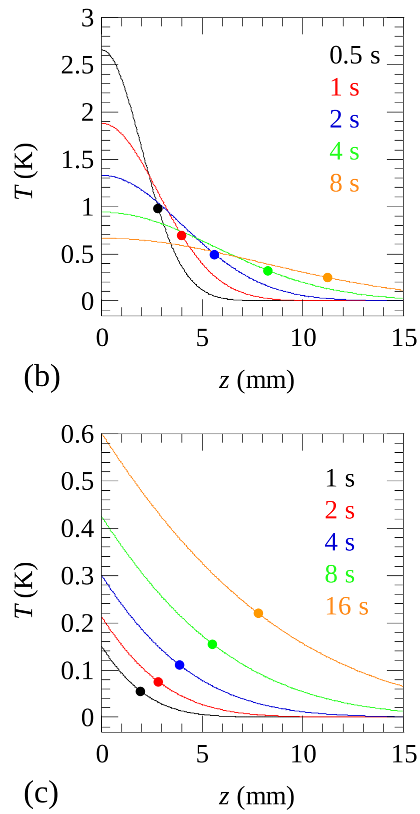




**FIGURE 2.** (a) Depth dependence of the temperature of a semi-infinite block of stainless steel whose free surface is put in contact with a thermal reservoir at 10 K. Six times are plotted. (b) The same when the steel block is illuminated by a brief flash lighting of 25 kJ/m$^2$. (c) The same when the steel block is illuminated by a continuous light beam of intensity 1 kW/m$^2$. Dots indicate the thermal diffusion length at each time.

As the thermal diffusion length depends on the square root of time, the speed of heat conduction is not constant, but it decreases with the square root of time:

$$v_{effective} = \frac{d\mu_{effective}}{dt} \approx \sqrt{\frac{D}{2t}}\,. \tag{19}$$

The quantity that governs the speed of heat conduction is the thermal diffusivity [6]. Typical values of this magnitude are 0.1-0.2 mm$^2$/s for thermal insulators as polymers, cork and wood; 0.5-2 mm$^2$/s for glasses and oxides; 10-30 mm$^2$/s for titanium, nickel, lead and steel; 100-200 mm$^2$/s for aluminium, copper, silver and gold; and $10^3$ mm$^2$/s for diamond, the bulk material with the highest thermal diffusivity [1]. Figure 3 represents the penetration depth of heat by conduction as a function of time according to Eq. (18). Five values of the thermal diffusivity, covering the whole range of solid materials, are plotted. We use logarithmic scale, where this relationship is linear with a slope of 0.5. This means that, for all kinds of materials, to increase the penetration depth by a factor of ten one has to wait for a time two orders of magnitude longer. This figure allows a quick estimation of the penetration depth of heat for any kind of material. It is worth mentioning that between the poorest thermal conductors (polymers) and the best one (diamond) the penetration depth changes a factor of a hundred. For instance, in the first second after stimulating the sample surface, heat travels 0.45 mm in a polymeric sample, but 4.5 cm in diamond (see Fig. 3).

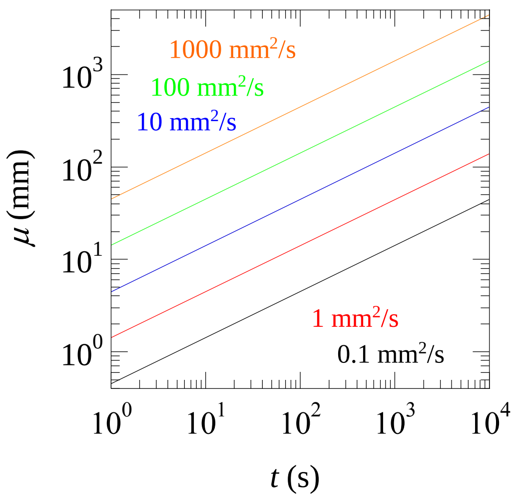



**FIGURE 3.** Penetration depth of heat as a function of time according to Eq. (17) in logarithmic scale. Five thermal diffusivities are considered.

## IV. CONCLUSIONS

The heat diffusion equation predicts an instantaneous temperature elevation at any position of the material after a local perturbation. Although this issue deserves fundamental research, it is not a constraint when studying heat conduction problems at a macroscopic level since the predicted temperature rise far away from the perturbation region is so small that it is non-measurable. However, the question of how deep heat propagates by conduction inside a solid sample still remains. In this paper, we have proposed a mathematical expression for the thermal diffusion length, which is valid for several configurations of surface stimulation: a thermal reservoir, a brief flash lighting and a continuous light beam. This expression allows lecturers and students to estimate the depth penetration, which depends on the thermal diffusivity and time. We expect this article to be useful for physics lecturers to attract the attention of physics students on heat conduction.

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