Equilibrium of four cubes on a balance: Exploring high-school students' answers and different ways of challenging their "fast thinking"

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Abstract

From the results of a previous research with Mexican students, it is known that Perelman's puzzle about the equilibrium of four cubes, with sides of 6 cm, 8 cm, 10 cm and 12 cm, activate in many students "fast thinking" based on the "illusion of linearity". The students think that, in the equilibrium configuration, 6-cm and 12-cm cubes should be on one plate while 8-cm and 10-cm cubes should be placed on the other plate. The puzzle was given to three groups of Slovenian high-school students (N = 67 students). After they provided their answers, students had three different ways to verify (to confirm or to challenge) these answers. The first group (N_1 = 24 students, age 15 years) was given correct answer as a report of a student who allegedly carried out the experiment at home. The second group (N_2 = 22 students, age 15 years) saw the photo of equilibrium situation. The third group (N_3 = 23 students, age 16 years) had a hands-on opportunity, with four cubes and a balance, to find the correct answer. The results show that the best way to change students' "fast thinking" is to provide hands-on activity. Telling the correct verbal answer or showing the photo of the correct configuration of the cubes on the balance are not very convincing way to challenge "fast thinking" of some students.

Keywords: Puzzle-based learning, fast thinking, illusion of linearity.

Resumen

A partir de los resultados de una investigación previa con estudiantes mexicanos, se sabe que el rompecabezas de Perelman sobre el equilibrio de cuatro cubos, con lados de 6 cm, 8 cm, 10 cm y 12 cm, activa en muchos estudiantes el "pensamiento rápido" basado en la "Ilusión de linealidad". Los estudiantes piensan que, en la configuración de equilibrio, los cubos de 6 cm y 12 cm deberían estar en un plato mientras que los cubos de 8 cm y 10 cm deberían colocarse en el otro plato. El rompecabezas se entregó a tres grupos de estudiantes de secundaria eslovenos (N = 67 estudiantes). Después de dar sus respuestas, los estudiantes tenían tres formas diferentes de verificar (confirmar o cuestionar) estas respuestas. El primer grupo (N1 = 24 estudiantes, 15 años de edad) recibió la respuesta correcta como un informe de un estudiante que supuestamente llevó a cabo el experimento en casa. El segundo grupo (N2 = 22 estudiantes, 15 años) vio la foto de la situación de equilibrio. El tercer grupo (N3 = 23 estudiantes, de 16 años de edad) tuvo una oportunidad práctica, con cuatro cubos y una balanza, para encontrar la respuesta correcta. Los resultados muestran que la mejor manera de cambiar el "pensamiento rápido" de los estudiantes es proporcionar actividades prácticas. Decir la respuesta verbal correcta o mostrar la foto de la configuración correcta de los cubos en la balanza no es una forma muy convincente de desafiar el "pensamiento rápido" de algunos estudiantes.

Palabras clave: aprendizaje basado en rompecabezas, pensamiento rápido, ilusión de linealidad.

I. INTRODUCTION

Quantitative, visual, logical, or manipulative puzzles have long been used as useful tasks in the study of human cognitive processes [1] and in books that teach general strategies to improve thinking, learning, and creativity [2] However, books and articles on how to systematically use puzzles to learn critical thinking and problem-solving skills have recently started to appear in the education of engineers [3, 4, 5].

Puzzle-based learning is considered effective in the systematic and timely development of such skills for the following reasons:

The puzzles are instructive because they illustrate useful (and powerful) rules for solving problems in a very fun way. The puzzles are fascinating and challenging.

Contrary to many textbook problems, puzzles are not "tied" to any chapter (as is the case with real-world problems).

It is possible to talk about different techniques (for example, simulation and optimization), disciplines (such as probability and statistics) or application areas (for example, programming and finance) and illustrate their meaning by discussing some simple puzzles. At the same time, students are aware that many of the findings are applicable in the broader context of real-world problem solving [3, p. XII].

It is important to emphasize that in recent years Microsoft, Google, and other high-tech companies, in their extremely strenuous job interviews, have used a large number of logical and mathematical puzzles and "impossible questions" [6, 7, 8] (Poundstone, 2003; Kador, 2005; Poundstone, 2012). "Interviews with puzzles" have become a new trend. From Wall Street to Silicon Valley, employers use tough questions to test a candidate's intelligence, imagination, and problem-solving ability. Those are the essential skills for survival and success in today's competitive global marketplace.

Managers looking for the most talented employees need to learn how to incorporate good (and unfamiliar) puzzles into the conversations that will lead to the search for the best candidates. Job seekers must figure out how to deal with "brain-exploding" problems and how to get a head start that could lead to a job for life. John Kador, a renowned expert in job interview questions, describes well the economic reasons for such a strategy:

"Using puzzles and riddles makes sense in companies that focus recruitment efforts more on what candidates can do in the future than what they have done in the past. These companies understand that in today's fast-paced world of global business, specific skills are of limited use because technology changes so rapidly. What is really needed, according to the interviewers, are curious, observant and resourceful candidates who embrace new challenges, demonstrate mental agility under stressful conditions, learn quickly, defend their thinking and demonstrate enthusiasm for impossible tasks. " [7, p. VI].

It should be noted that the "Cognitive Reflection Test" [9], frequently used to detect "fast" and "slow" thinkers [10] among economics students, consists of three famous mathematical puzzles. Some research indicates that "fast" thinkers (those who give three wrong answers on the test) are the ones who make mistakes more frequently in economic tasks that require adequate information processing and decision-making [11].

All these educational trends and economic facts require increasing the presence of mathematical and quantitative puzzles in the teaching of mathematics, physics and engineering through the development, implementation and evaluation of new designs for its didactic uses.

II. PERELMAN'S PUZZLE "FOUR CUBES"

Yakov Isidorovich Perelman (Fig. 1) was the most famous Russian and Soviet science divulgator and writer of many books for the general public on physics [12], astronomy [13], and mathematics [14].

Perelman was very interested in puzzles and was the first to present in Russian many of puzzles written and published previously by legendary puzzle-makers: Englishman Henry Ernest Dudeney and American Sam Loyd.

The puzzle "Four cubes", that was used in this research, has seen the light in 1935 in Russian. Its English version appeared in the book "Fun with Maths and Physics", a collection of Perelman's problems, first published in 1984 and, in second printing in 1988 [15].

FIGURE 1. Yakov Isidorovich Perelman (December 4, 1882–March 16, 1942).

Its formulation was:

Four Cubes

Four solid cubes of the same material have different heights: 6 cm, 8 cm, 10 cm, and 12 cm (Fig. 299). Arrange them on the pans of a balance for it to be in equilibrium. [15, p. 341]

The illustration mentioned in the puzzle's formulation is given in the Fig. 2:

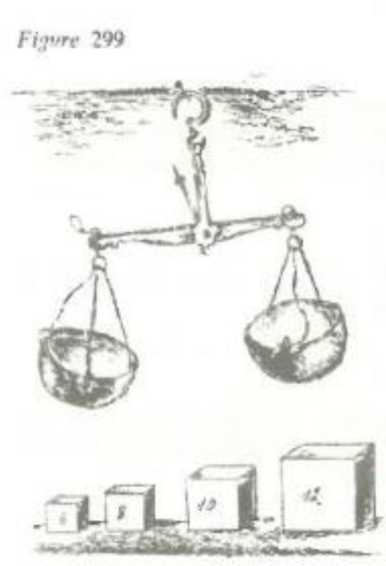

FIGURE 2. The cubes and the balance mentioned in the Perelman's puzzle.

The given answer was:

"We must place three smaller cubes on one pan, and the largest one on the other. It's easily verified that the balance will be in equilibrium. Let's show that the total volume of the three smaller cubes equals that of the largest one. This follows from the relationship

$$6^3 + 8^3 + 10^3 = 12^3,$$

i.e.

$$216 + 512 + 1{,}000 = 1{,}1728.\text{” [15, p. 347]}$$

Two technical pedantic comments on the answer are necessary.

1) The condition of equilibrium “in terms of volumes”, is a consequence of the equilibrium condition “in terms of masses” which holds because all cubes have the same density.

2) The volumes should have been expressed using explicitly their corresponding physical units (cm^3).

III. THE PUZZLE AND THE “ILUSION OF LINEARITY”: RESULTS OF A PREVIOUS STUDY IN MEXICO

For the purpose of this study, a cognitive comment is more important. Neither Perelman in 1935 nor the editor of the book in 1988 has noted that the puzzle “Four cubes” would likely activate in many persons “fast thinking” [10], leading to an erroneous answer:

The balance is in equilibrium when 6-cm and 12-cm cubes are on one pan and 8-cm and 10-cm are on the other.

This is very surprising taking into account that Perelman, in the answers for many puzzles, mentioned “fast answer”. One example is the puzzle “Binding”:

“Here is an insidious problem. A bound book cost 2 roubles 50 kopecks. The book is 2 roubles more expensive than the binding. How much does the binding cost?” [15, p. 276].

Description of erroneous (“fast thinking”) approach to solving “Binding” puzzle is given before presentation of the correct (“slow thinking”) answer:

“The off-the-cuff answer is usually: the binding costs 50 kopecks. But then the book would cost 2 roubles, i. e. it would be only 1 rouble 50 kopecks more expensive than the binding. The true answer is: the binding costs 25 kopecks, the book 2 roubles 25 kopeck with the result that the book costs 2 roubles more than the binding. [15, p. 282].

A good example of “fast thinking” is well-documented experimental fact that students resort to inappropriate use of the “rule of three” in solving problems that refer to situations in which the relationship between the variables is not linear [16, 17, 18, 19, 20). This phenomenon has been called "the illusion of linearity."

It is especially prevalent in geometry problems [21, 22]. When students are given the question “If the edge of a cube is 3 cm, its volume is 27 cm^3. What will be the volume of a cube whose edge is 6 cm?”, many of them would give wrong, “fast thinking” answer: “54 cm^3”. They believe in correctness of the rule “double-edged cube has doubled volume”. Although they very likely know the formula for the volume of cube of side a ($V = a^3$), they don’t use in “fast thinking” about the answer, but rather suppose a linear relationship between volume of the cube and its side.

It was interesting to explore if students use this reasoning in dealing with a more complex situation related to equilibrium of four cubes on a balance when the question about volume of cubes is not explicitly asked.

Initial study was designed and carried out with 277 Mexican high-school students [23]. In the first part of task, each student was asked (1) to draw the configuration of the four cubes in the equilibrium situation and (2) to argue why she or he believes in the correctness of the drawing. Only 6 students (2.1 %) were able to present a correct drawing and to give an acceptable argument for it.

A typical incorrect drawing can be seen in the Fig. 3.

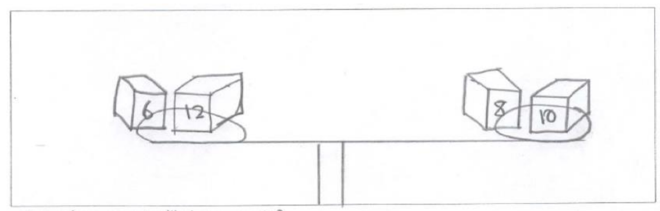

FIGURE 3. A typical incorrect drawing for the equilibrium of four cubes.

The argument correctness had frequently two parts: (1) this is a “just” distribution of the cubes (two on each pan) and (2) the sum of the cube sides on both pans is equal to 18.

In the second part of the task, three sub-groups of students were given correct answers (three smaller cubes on one pan and the biggest cube on the other pan) but allegedly with different “answer authors”: (a) a high-school student, (b) their math teachers and (c) a university professor of mathematics. Students had options evaluate correctness of that answer: (i) It is correct; (ii) It is incorrect; (iii) It can be either correct or incorrect.

Surprisingly, 237 students who gave incorrect answer evaluated the given answer as “incorrect”, no matter who was alleged its author. Students’ low disposition to change their answer when other students present a different answer was expected. It was as a surprise to find out that students kept confidence in their “fast thinking” answer even if the different (in this case, the correct) answer came from someone who, according to common sense, possess much more mathematical knowledge. This result shows how deeply “illusion of linearity” (as basis for “fast thinking”) is rooted in the students.

IV. STUDENTS’ PERFORMANCES IN “FOUR CUBES” PUZZLE IN SLOVENIA

The study whose results are reported in this article was carried a few years ago in Slovenia with three groups of Slovenian high-school students (N = 67 students). As in Mexican study, students had two-part tasks. The first part was the question about the equilibrium configuration of the cube on the pans of balance. In that part 18 students (26.8 %) were able to give the correct answer.

After they provided their answers, the students had three different ways to verify (to confirm or to challenge) these answers.

The first group (N_1 = 24 students, age 15 years) was given correct answer as a verbal report of a student who allegedly carried out the experiment at home.

The second group (N_2 = 22 students, age 15 years) saw a photo of equilibrium situation.

The third group (N_3 = 23 students, age 16 years) had a hands-on opportunity, with four cubes and a balance, to find the correct answer.

In what follows, more details about students' performances in each group are presented.

A. The first group: The correct answer given as an alleged student's verbal report

This part of the study was, in fact, a replica of corresponding part in Mexican study.

The performances were the following:

Nine students (37.5 %) predicted and argued correctly under what conditions the balance will be in equilibrium. They all wrote the volume of the 12 cm cube is equal to the sum of volumes of 6 cm, 8 cm and 10 cm cubes.

Assuming the density of all cubes is the same, the masses on both arms of the balance are the same and the balance is in equilibrium. They all wrote the story of the pupil in the task only confirmed their thinking.

Fifteen students (62.5 %) gave an incorrect answer. Their arguments were:

Three of them argued the sum of torques on left arm of the balance must be the same as the sum of torques on right side but they didn't make the connection between the mass of the cube and the side of the cub.

Eleven of them argued the sum of the sides on both arms of the balance must be the same.

One student had no argument.

Six students changed their minds and accepted the verbal report of the student as correct. They all recognized a linear relationship between the mass and the volume instead between the mass and the side.

Eight students refused to change their mind and wrote that the verbal report can't be correct because the sum of the masses of the three smaller cubes is certainly bigger than the mass of the cube with the 12 cm side.

One student wrote he is confused but didn't wrote any argumentation.

As in Mexico, more than 50 % of Slovenian students are reluctant to change their "fast thinking" if the correct answer comes from a student.

The forms of the second part of the task for the next two groups (opportunity to change the "fast thinking" answer) were new.

B. The second group: The photo of the correct answer was given

In the second part, students in this group were given the photo of equilibrium configuration for the cubes on the balance (Fig.4)

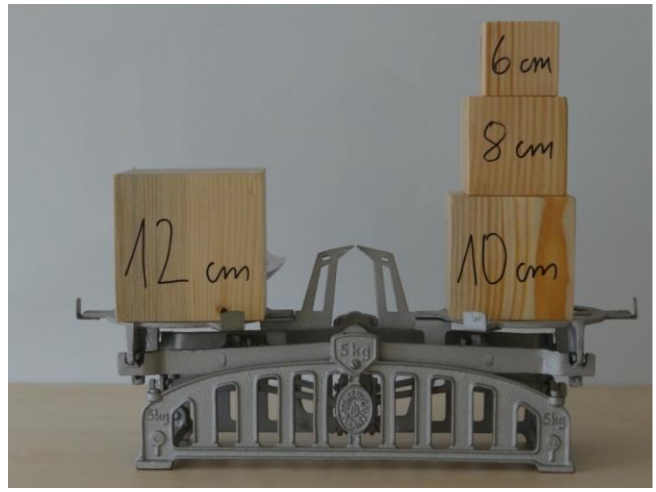

FIGURE 4. The photo of equilibrium configuration for the cubes on the balance.

The performances of the students in two tasks were the following ones:

Four students (18 %) predicted and argued correctly under which conditions the balance will be in equilibrium. They all wrote the volume of the 12 cm cube is equal to the sum of volumes of 6 cm, 8 cm and 10 cm cubes.

Assuming the density of all cubes is the same, the masses on both arms of the balance are the same and the balance is in equilibrium.

They all wrote the photo only confirmed their thinking.

Eighteen students (82 %) gave the incorrect answer.

Six of them argued: the sum of torques on left arm of the balance must be the same as the sum of torques on right side, but they didn't make the connection between the mass of the cube and the side of the cube

Two of the wrote correctly that the masses on both sides of the balance must be the same, but were unable find the connection between the mass and the side of the cube.

One student had no argument.

Nine students argued the sum of the sides of cubes on both arms of the balance must be the same.

In the subgroup with incorrect answer, eight students wrote that the photo was correct, recognizing that they made an incorrect prediction. They all recognized linear connection between the mass and the volume instead the mass and the side.

Four students wrote they are confused but argued anyway that the sum of the volumes on both arms of the balance must be the same.

Five students admitted they are confused, saying that the distribution of the cubes in the picture "is not logical".

One student wrote:

"The sum of the masses of the three smaller cubes is SURELY bigger than the mass of the cube with the 12 cm side. Being so, the picture is wrong and probably mounted!"

In this group, for six students who initially had wrong answer, the photo of the correct configuration was not a convincing argument for changing their minds.

C. The third group: The students were given a hands-on opportunity to find the correct answer

In the second part, students in this group were given the four cubes and balance (Fig. 5) and were invited to find the right answer through a hands-on activity.

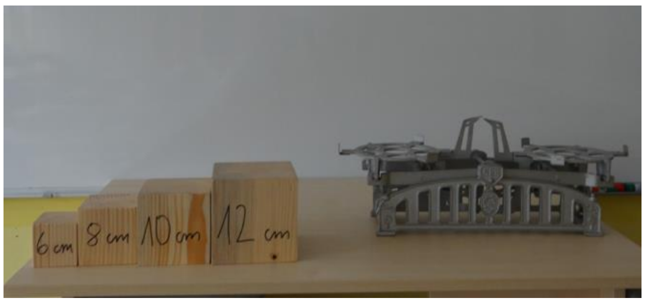

FIGURE 5. Four cubes and the balance given to students for hands-on activity to find the equilibrium configuration.

Their performances in both tasks were the following:

Five students (22 %) predicted and argued correctly under what conditions the balance will be in equilibrium. They all wrote that volume of the 12 cm cube is equal to the sum of volumes of 6 cm, 8 cm and 10 cm cubes.
Assuming the density of all cubes is the same, the masses on both arms of the balance are the same and the balance is in equilibrium.

All students in this subgroup expressed quite happily that the experiment (hands-on activity) confirmed their initial "theoretical" thinking.

Eighteen students (78 %) predicted incorrectly the configuration of equilibrium.

Seventeen of them argued the sum of the sides on both arms of the balance must be the same.

One student had no argument and was just guessing.

After the students carried out experimental exploration, sixteen of them were convinced that the correct configuration was:

"the 12-cm cube on one arm and and other three cubes on the other arm"

All of them recognized linear relationship between the mass and the volume of the cube.

One student wrote he was confused and didn't understand what is going on.

In addition, one more student was confused, too, He wrote: "There is SOMETHING WRONG with the experiment!"

VI. CONCLUSIONS, IMPLICATIONS FOR TEACHING AND POSSIBLE FURTHER RESEARCH

This initial qualitative study explored:

(1) how three group of Slovenian high-school students answered the question in the Perelman's puzzle "Four cubes"; and

(2) what were their performances in three different "encounters" with the correct answer (a verbal student's report of an experiment carried out at home, a photo of the correct answer and a hands-on experimental exploration).

The results of this study show that, for almost all students, the best way to convert "fast thinking" into a "slow thinking" is through unguided hands-on experimentation.

The next-to-best way of changing students' initial wrong answer is by presenting the photo of equilibrium configuration of the cubes. Nevertheless, the number of students reluctant to change is in this mode bigger than in the case of the hands-on exploration.

An inadequate way to change students' initial "fast thinking" is to just give them the correct verbal answer:

> 12-cm cube should be on the one pan and three other three cubes should be on the other pan.

Generally speaking, in puzzle-based learning or in conceptually challenging topics for which students have alternative, common-sense conceptions, teachers should give all opportunities to detect and define their conceptual and procedural errors ("fast thinking") and to correct them, moving toward correct answers ("slow thinking") through group collaboration. "Productive failures" are a good way to learn better [24].

A weak aspect of both studies was the fact that the correct verbal answer was given to students without any argument, hoping that will be able to grasp its conceptual basis. Obviously, that "slow thinking" analysis was too demanding task for many students.

Being so, in future studies, it would be worth of exploring whether and how much different forms of arguments (conceptual and mathematical), supporting the correct answer, would be able to help students get free of their erroneous "fast thinking".

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