

New approximate solutions of the Schrödinger equation with improved energy-dependent screened Coulomb potential in 3D-NCPS symmetries

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Abstract

By using the generalized Bopp's shift method, conventional perturbation theory, and the Greene-Aldrich approximation to handle the centrifugal term, we have studied improved energy-dependent screened Coulomb potential model in three-dimensional non-commutative phase-space in the framework of deformed Schrödinger equation (DSE). We discovered the new energy eigenvalues of neutral and Hydrogenic atoms. The obtained eigenfunctions were a function of the discrete atomic quantum numbers (j, l, s , and m), the gas state, the non-commutativity parameters ($\theta, \bar{\theta}$), and the dimensional parameters of the standard energy-dependent screened Coulomb potential (g, A, α), which shows the depth of the potential, the energy slope parameter, and the potential range. The associated Hamiltonian operator, combining the standard energy-dependent screened Coulomb potential Hamiltonian operator with three additional operators, the perturbed spin-orbit interaction, we have confirmed the new modified Zeeman operator and the perturbative Fermi gas. By altering a few potential parameters, we explore two exceptional situations to demonstrate the precision of our results. We think this is the development of hydrogen atom devices and atomic physics.

Keywords: Schrödinger equation, Energy-dependent screened Coulomb Potential, Non-commutative phase-space, Generalized Bopp's shift method, Star product, Non-commutative phase-space.

Resumen

Utilizando el método de desplazamiento de Bopp generalizado, la teoría de perturbaciones convencional y la aproximación de Greene-Aldrich para manejar el término centrífugo, hemos estudiado un modelo mejorado de potencial de Coulomb apantallado dependiente de la energía en un espacio de fases no conmutativo tridimensional en el marco de la ecuación de Schrödinger deformada (DSE). Descubrimos los nuevos autovalores de energía de átomos neutros e hidrogénicos. Las funciones propias obtenidas fueron una función de los números cuánticos atómicos discretos (j, l, s y m), el estado gaseoso, los parámetros de no conmutatividad ($\theta, \bar{\theta}$) y los parámetros dimensionales del potencial de Coulomb apantallado dependiente de la energía estándar (g, A, α), que muestra la profundidad del potencial, el parámetro de pendiente de energía y el rango del potencial. El operador hamiltoniano asociado, que combina el operador hamiltoniano del potencial de Coulomb apantallado dependiente de la energía estándar con tres operadores adicionales, la interacción espín-órbita perturbada, hemos confirmado el nuevo operador Zeeman modificado y el gas de Fermi perturbativo. Al modificar algunos parámetros potenciales, exploramos dos situaciones excepcionales para demostrar la precisión de nuestros resultados. Creemos que se trata del desarrollo de dispositivos basados en átomos de hidrógeno y de la física atómica.

Palabras clave: Ecuación de Schrödinger, Potencial de Coulomb apantallado dependiente de la energía, Espacio de fases no conmutativo, Método de desplazamiento de Bopp generalizado, Producto estrella, Espacio de fases no conmutativo.

I. INTRODUCTION

It is known that exact solutions of the non-relativistic state of the Schrödinger equation (SE), Klein-Gordon equation (KGE), and Dirac equation (DE) were achieved exclusively in the two aspects of the harmonic oscillator and the hydrogen atom. In the past years, many researchers have investigated SE, KGE, and DE using many methods, including the Nikiforov-Uvarov method, asymptotic iteration method, supersymmetric quantum mechanics, factorization method in quantum mechanics, and functional analysis method, to solve many problems for example, the Hulthén potential [1], the

Eckart potential [2], the hyperbolic potential [3], the Poschel-Teller potential [4], and the generalized hyperbolic potential model [5]. The Yukawa potential is considered one of the most important potentials, which attracted researchers to study it because it contains profound applications in many fields. This potential, also known as the static screened Coulomb, the Debye-Hückel, or the Thomas-Fermi [6], is an approximate solvable potential often used to compute bound-state normalizations and energy levels of neutral atoms [7, 8, 9, 10]. These potentials have received considerable attention since the early days of quantum mechanics because of the wide range of applications previously mentioned, they have been considered one of the oldest pieces of knowledge since

1935 by researchers and the latest interest in the microscopic scale [10, 11, 12, 13]. Furthermore, this potential also plays an important role in plasma physics, known as the Debye-Huckel potential. In addition to these obvious physical applications, this potential, together with Hulthen's and the exponential potential, plays an essential role as a good test case in potential scattering studies [14, 15, 16, 17]. Furthermore, it is used to compute bound-state normalizations and energy levels of neutral atoms, and it is also used in dusty or complex plasma and colloidal suspensions [18]. The study of the Yukawa potential still receives excellent attention from many authors [19, 20, 21, 22]. In 2017 and 2020, we studied the Yukawa potential and the perturbed Yukawa potential in non-commutative, non-relativistic quantum mechanics and relativistic contexts using Bopp's shift method parameter for one-electron atoms [23, 24]. Recently, we have studied hydrogen-like atoms (or hydrogenic atoms) such as $(\text{He}^+, \text{Li}^{2+} \text{ and } \text{Be}^{3+})$ under an improved trigonometric Rosen-Morse potential model in the 3D-NCPS Symmetries [25]. In particular, the Yukawa potential describes the interactions of hydrogen-like atoms [24]. Many researchers in the fields of physics and chemistry have also devoted great attention to studying the non-relativistic and relativistic dynamical quantum systems of the energy dependence of variant potentials, for example, the energy-dependent screened Coulomb potential [26]. This work, motivated by several recent studies such as the non-renormalizable standard model, string theory, quantum gravity, and NCQM, has attracted much attention [27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. In 1930, Heisenberg introduced the idea of non-commutativity, which was formalized by Snyder in 1947. We want to extend the study in Ref. 26 to the case of NR-NCQM to explore the possibility of finding other new applications and more profound interpretations of the subatomic scales under improved energy-dependent screened Coulomb potential. The non-relativistic energy levels of hydrogen atoms and neutral atoms such as $(^{22}\text{Na}, ^{12}\text{Ca}, ^{158}\text{Au})$ atoms, which interacted with improved energy-dependent screened Coulomb potentials with arbitrary angular momenta in the context of 3D-NCPS symmetries, have not yet been studied. The major objective of the present paper is to obtain approximate solutions of the DSE with an improved energy-dependent screened Coulomb potential in 3D-NCPS symmetries using the improved approximation scheme to the centrifugal term for $l \neq 0$ states and the generalized Bopp shift method, in addition to the standard perturbation theory. We believe that no researcher has realized this research so far. The new symmetries have a twofold effect, the energy-dependent directly influences the first screened Coulomb potential $V_{esc}(r)$, which takes on a new form (we depend on the system of natural units in this research $c = \hbar = 1$):

$$V_{nc}^{esc}(r) = V_{esc}(r) - A(1 + gE_{nl}) \frac{\exp(-ar)}{2r^2} \left(\alpha + \frac{1}{r} \right) \overleftrightarrow{\mathbf{L}} \mathbf{0}. \quad (1)$$

The first term [26] $V_{esc}(r)$ is given by:

$$V_{esc}(r) = -\frac{A(1+gE_{nl})\exp(-ar)}{r}, \quad (2)$$

which defines the energy-dependent screened Coulomb potential. Here A and g are the potential depth, and the energy slope parameter is the range of the potential and the distance between the two particles r . As for the second part of the effect of topological defects, it is of a phased nature, that is, it is on the kinetic energy known in the literature to become as follows:

$$-\frac{\Delta}{2\mu} \rightarrow -\frac{\Delta_{nc}}{2\mu} = -\frac{\Delta}{2\mu} + \frac{\overleftrightarrow{\mathbf{L}}}{2\mu}. \quad (3)$$

The new couplings $(\overleftrightarrow{\mathbf{L}} \mathbf{0}$ and $\overleftrightarrow{\mathbf{0}} \mathbf{L})$ will be defined in Sect. 3. While (Δ_{nc}, Δ) are the Laplacian operators in 3D-NCPS symmetries and the usual quantum mechanics. Furthermore, the second part of Eq. (1) denotes the influence of the non-commutativity proprieties on the standard energy-dependent screened Coulomb potential. Moreover $(-\frac{\overleftrightarrow{\mathbf{L}}}{2\mu})$ is the change applied to the operator of kinetic energy due to the influence of the non-commutativity proprieties. The current work will study the effect of non-commutativity properties on the potential and the kinetic energy using the generalized Bopp's shift method (see the third section). The structure of non-relativistic non-commutative quantum mechanics based on NC time-independent canonical commutation relations in Schrödinger picture (SP), Heisenberg picture (HP), and interactions picture (IP), respectively, as follows [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]:

$$[\hat{x}^{\mu*}, \hat{p}^{\nu}] = [\hat{x}^{\mu}(t)^*, \hat{p}^{\nu}(t)] = [\hat{x}_l^{\mu}(t)^*, \hat{p}_l^{\nu}(t)] = i\hbar^{eff} \delta^{\mu\nu}, \quad (4)$$

$$[\hat{x}^{\mu*}, \hat{x}^{\nu}] = [\hat{x}^{\mu}(t)^*, \hat{x}^{\nu}(t)] = [\hat{x}_l^{\mu}(t)^*, \hat{x}_l^{\nu}(t)] = i\theta^{\mu\nu}, \quad (5)$$

and

$$[\hat{p}^{\mu*}, \hat{p}^{\nu}] = [\hat{p}^{\mu}(t)^*, \hat{p}^{\nu}(t)] = [\hat{p}_l^{\mu}(t)(t)^*, \hat{p}_l^{\nu}(t)] = i\bar{\theta}^{\mu\nu}, \quad (6)$$

where the indices $\mu, \nu \equiv \overline{1,3}$, $\hbar^{eff} = \hbar \left(1 + \frac{\theta\bar{\theta}}{4\hbar^2} \right)$ is the effective Planck constant. The two infinitesimal parameters $(\theta^{\mu\nu}, \bar{\theta}^{\mu\nu}) \equiv \varepsilon^{\mu\nu}(\theta, \bar{\theta})$ (If compared with energy) are present two antisymmetric elements real matrices with dimensions of the square length and square momentum, respectively, and the notion $\delta^{\mu\nu}$ denote to the Kronecker symbol. Furthermore, $(*)$ denote the star product that is generalized between two arbitrary functions $(f, g)(x, p)$ to the new form $(\hat{f}\hat{g})(\hat{x}, \hat{p}) \equiv (f * g)(x, p)$ in 3D-NCPS symmetries as follows [48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58]:

$$(f * g)(x, p) \approx \left(fg - \frac{i}{2} \theta^{\mu\nu} \frac{\partial f}{\partial x^{\mu}} \frac{\partial g}{\partial x^{\nu}} - \frac{i}{2} \bar{\theta}^{\mu\nu} \frac{\partial f}{\partial p^{\mu}} \frac{\partial g}{\partial p^{\nu}} \right) (x, p). \quad (7)$$

This permuted us to construct a scale of two space and phase cells with volumes $\mathbf{I}_{ncs}^3 \equiv \bar{\theta}^3$ and $\mathbf{I}_{ncp}^3 \equiv \theta^{-3/2}$, respectively. On the other hand, Eq. (7) enables us to satisfy the

hypothesized algebra in Eqs. (4), (5) and (6). The second and third terms in the above equation are the effects of (space-space) and (phase-phase) non-commutativity properties, respectively. This means that the principle of uncertainty for Heisenberg is generalized to include other new uncertainty:

$$|\Delta\hat{x}^\mu\Delta\hat{x}^\nu| \geq \frac{|\theta^{\mu\nu}|}{2} \quad \text{and} \quad |\Delta\hat{p}^\mu\Delta\hat{p}^\nu| \geq \frac{|\bar{\theta}^{\mu\nu}|}{2}, \quad (8)$$

related to the positions $(\hat{x}^\mu, \hat{x}^\nu)$ and corresponding momenta $(\hat{p}^\mu, \hat{p}^\nu)$, respectively, in addition to the usual uncertainty known in the literature:

$$|\Delta x \Delta p| \geq \frac{\delta^{\mu\nu} \hbar}{2} \Rightarrow |\Delta\hat{x}^\mu\Delta\hat{p}^\nu| \geq \frac{\delta^{\mu\nu} \hbar^{eff}}{2}, \quad (9)$$

of $(\hat{x}^\mu, \hat{p}^\nu)$ coordinates which are known in the literature. However, the operators $\hat{\xi}_H^\mu(t) = (\hat{x}^\mu \vee \hat{p}^\mu)(t)$ and $\hat{\xi}_I^\mu(t) = (\hat{x}_I^\mu \vee \hat{p}_I^\mu)(t)$ in both two representations (HP and IP), respectively, depending on the corresponding usual operator $\hat{\xi}_{iS} = \hat{x}_i \vee \hat{p}_i$, in SP which known in the literature, as follows:

$$\begin{aligned} \hat{\xi}_H^\mu(t) &= \exp(i\hat{H}_{esc}T) \hat{\xi}_S^\mu(t) \exp(-i\hat{H}_{esc}T) \Rightarrow \\ \hat{\xi}_H^\mu(t) &= \exp(i\hat{H}_{nc}^{esc}T) * \hat{\xi}_S^\mu(t) * \exp(-i\hat{H}_{nc}^{esc}T), \end{aligned} \quad (10)$$

and

$$\begin{aligned} \hat{\xi}_I^\mu(t) &= \exp(i\hat{H}_o^{esc}T) \hat{\xi}_S^\mu(t) \exp(-i\hat{H}_o^{esc}T) \\ \Rightarrow \hat{\xi}_I^\mu(t) &= \exp(i\hat{H}_{nc}^{esc}T) * \hat{\xi}_S^\mu(t) * \exp(-i\hat{H}_{nc}^{esc}T). \end{aligned} \quad (11)$$

Here $T \equiv t - t_0$, $\hat{\xi}_S^\mu = x^\mu \vee p^\mu$, $\hat{\xi}_H^\mu(t) = (x^\mu \vee p^\mu)(t)$ and $\hat{\xi}_I^\mu(t) = (x_I^\mu \vee p_I^\mu)(t)$ are just the physical values in the usual QM symmetries. The operators $(\hat{H}_{nc}^{esc}, \hat{H}_{nc}^{esc})$ and $(\hat{H}_{esc}, \hat{H}_o^{esc})$ are the standard and free quantum Hamiltonian for energy-dependent screened Coulomb in the are the corresponding Hamiltonian operators for improved energy-dependent screened Coulomb in 3D-NCPS and QM symmetries, respectively. The motion equations of dynamic systems $\frac{d\hat{\xi}_H^\mu(t)}{dt}$ in QM will change to the form $\frac{d\hat{\xi}_H^\mu(t)}{dt}$ of the symmetries of 3D-NCPS as follows:

$$\begin{aligned} \frac{d\hat{\xi}_H^\mu(t)}{dt} &= -i[\hat{\xi}_H^\mu(t), \hat{H}_{yp}] + \frac{\partial\hat{\xi}_H^\mu(t)}{\partial t} \Rightarrow \\ \frac{d\hat{\xi}_H^\mu(t)}{dt} &= -i[\hat{\xi}_H^\mu(t), \hat{H}_{nc}^{yp}] + \frac{\partial\hat{\xi}_H^\mu(t)}{\partial t}. \end{aligned} \quad (12)$$

The structure of the current paper is as follows: The ordinary SE with energy-dependent screened Coulomb based on Ref. 26 is briefly discussed in the next section. In Section III, we study the DSE using the generalized Bopp shift method for an improved energy-dependent screened Coulomb potential. Then, using standard perturbation theory, we find the corrected quantum spectrum of $(n, l, m)^{th}$ excited levels induced with the help of spin-orbit interaction within the context of the local 3D-NCPS symmetries. Then, we calculate the magnetic and Fermi gas spectrum with

enhanced energy-dependent screened Coulomb. In Section IV, we continue discussing the global spectrum, the related NC Hamiltonian operator, and the energy levels of the neutral atoms (^{22}Na , ^{12}Ca , ^{158}Au) and hydrogen-like atoms (He^+ , Li^{2+} , B^{3+}) for an improved energy-dependent screened Coulomb potential. Finally, we summarize our work and give concluding remarks in Section V.

II. A BRIEF REVIEW OF THE EIGEN-FUNCTIONS AND THE ENERGY EIGEN-VALUES FOR STANDARD ENERGY DEPENDED SCREENED COULOMB POTENTIAL IN 3D-QM SYMMETRIES

The study of the SE for the energy-dependent screened Coulomb potential $V(r, E_{nl})$ in ordinary non-relativistic quantum mechanics is an essential source of understanding of this system in the symmetries of 3D-NCPS, we will devote this section to this purpose. Ref. [26] gives the version of the energy-dependent screened Coulomb potential determined in Eq. (2). Since the energy-dependent screened Coulomb potential has spherical symmetry, allowing the solutions of the time-independent Schrödinger equation of the known form $\Psi(r, \theta, \phi) = \frac{R_{nl}(r)}{r} Y_l^m(\theta, \phi)$, to separate the radial $R_{nl}(r)$ and angular $Y_l^m(\theta, \phi)$ parts of the wave function. Thus, the radial part becomes the following.

$$\frac{d^2 R_{nl}(r)}{dr^2} + 2\mu (E_{nl} - V_{eff}^{esc}(r, E_{nl})) R_{nl}(r) = 0 \quad (13)$$

Here

$$V_{eff}^{esc}(r, E_{nl}) \equiv -\frac{A(1+gE_{nl}) \exp(-\alpha r)}{r} + \frac{l(l+1)}{2\mu r^2} \quad (14)$$

Here $V_{eff}^{esc}(r, E_{nl})$ is the effective potential, μ is the reduced mass, E_{nl} are the eigenvalues of the energy-dependent screened Coulomb potential model, while n and l are the radial and orbital angular momentum quantum numbers. Ref. [26] gives the complete wave function as a function of the Jacobi polynomial and the spherical harmonic functions:

$$\Psi(r, \theta, \phi) = B_{nl} \frac{z^{\sqrt{\epsilon_{nl}^2}}}{r} (1-z)^{G_l} P_n \left(2\sqrt{\epsilon_{nl}^2} z^{2G_l-1} \right) (1-2z) Y_l^m(\theta, \phi), \quad (15)$$

with

$$\left. \begin{aligned} z &= \exp(-2\alpha r), \\ \epsilon_{nl}^2 &= -\left(\frac{\mu E_{nl}}{2\alpha^2} - \frac{l(l+1)}{12} \right), \\ G_l &\equiv 1/2 + \sqrt{1/4 + l(l+1)} \end{aligned} \right\} \quad (16)$$

while B_{nl} is the normalization constant. Therefore, Ref. [26] gives the discrete energy eigenvalues of the energy-dependent screened Coulomb potential as a function of the

principal quantum number and the angular momentum quantum number l :

$$E_{nl} = -\frac{\alpha^2}{2\mu} \left[\left(n + 1/2 + \sqrt{1/4 + l(l+1)} \right)^2 - \frac{l(l+1)}{3} \right] \quad (17)$$

III. METHOD AND THEORETICAL APPROACH

A. Solution of DSE for the improved energy-dependent screened Coulomb potential

In this subsection, we will provide an overview or a brief introduction to the improved energy-dependent screened Coulomb potential $V_{nc}^{esc}(r, E_{nl})$, in 3D-NCPS symmetries. To perform this task in the physical form of DSE, it is necessary to replace ordinary three-dimensional Hamiltonian operators $\hat{H}_{esc}(p_\mu, x_\mu)$, the ordinary complex wave function $\Psi(\vec{r})$, and E_{nl} (in QM-symmetries) with three Hamiltonian operators $\hat{H}_{nc}^{esc}(\hat{p}_\mu, \hat{x}_\mu)$, the complex wave function $\Psi(\hat{r})$, and new values E_{nc}^{esc} , respectively (in 3D-NCPS symmetries). Furthermore, replacing the ordinary product with a star product (*), allows us to construct the DSE in 3D-NCPS symmetries as [40, 41, 42, 43, 44]:

$$\hat{H}_{nc}^{esc}(\hat{p}_\mu, \hat{x}_\mu) * \Psi(\hat{r}) = E_{nc}^{esc} \Psi(\hat{r}). \quad (18)$$

Which can be written in the equivalent form:

$$\frac{d^2 R_{nl}(r)}{dr^2} + 2\mu(E_{nl} - V_{eff}^{esc}(r, E_{nl})) * R_{nl}(r) = 0. \quad (19)$$

We briefly introduce the generalized Bopp's shift method so that the unfamiliar reader has no problem with the mathematical tool. This method is widely used in 3D-NCPS, and the solutions enable us to explore an effective way to obtain improved potential in 3D-NCPS, which is based on the following commutators [33, 34, 35, 36, 37, 38, 39]:

$$\begin{cases} [\hat{x}_\mu, \hat{x}_\nu] = [\hat{x}_\mu(t), \hat{x}_\nu(t)] = i\theta_{\mu\nu}, \\ [\hat{p}_\mu, \hat{p}_\nu] = [\hat{p}_\mu(t), \hat{p}_\nu(t)] = i\bar{\theta}_{\mu\nu}. \end{cases} \quad (20)$$

The generalized positions and momentum coordinates $(\hat{x}_\mu, \hat{p}_\mu)$ in 3D-NCPS are dependent on corresponding usual generalized positions and momentum coordinates (x_μ, p_μ) in QM by the following, respectively [53, 54, 55, 56, 57, 58, 59]:

$$(x_\mu, p_\mu) \Rightarrow (\hat{x}_\mu, \hat{p}_\mu) = \left(x_\mu - \frac{\theta^{\mu\nu}}{2} p_\nu, p_\mu + \frac{\bar{\theta}^{\mu\nu}}{2} x_\nu \right). \quad (21)$$

It is worth mentioning that the physicist Fritz Bopp was the first to consider pseudo-differential operators obtained from *Lat. Am. J. Phys. Educ. Vol. 18, No. 2, June 2024*

a symbol by the quantization rules $(x, p) \rightarrow (x - \frac{i}{2} \frac{\partial}{\partial p}, p + \frac{i}{2} \frac{\partial}{\partial x})$ instead of the usual correspondence $(x, p) \rightarrow (x, \frac{i}{2} \frac{\partial}{\partial x})$ [56, 60, 61]. The above equation allows us to obtain the two operators (\hat{r}^2 and \hat{p}^2) in 3D-NCPS symmetries [58, 59]:

$$\underbrace{(r^2, p^2)}_{\text{QM}} \Rightarrow \underbrace{(\hat{r}^2, \hat{p}^2)}_{\text{3D-NCPS}} = \left(r^2 - \vec{\mathbf{L}}\vec{\Theta}, p^2 + \vec{\mathbf{L}}\vec{\Theta} \right). \quad (22)$$

The two couplings ($\vec{\mathbf{L}}\vec{\Theta}$ and $\vec{\mathbf{L}}\vec{\bar{\Theta}}$) are $(L_x\theta_{12} + L_y\theta_{23} + L_z\theta_{13})$ and $(L_x\bar{\theta}_{12} + L_y\bar{\theta}_{23} + L_z\bar{\theta}_{13})$, respectively and $(L_x, L_y$ and $L_z)$ are the three components of the angular momentum operator $\vec{\mathbf{L}}$ while $\Theta_{\mu\nu} = \theta_{\mu\nu}/2$ (see refs.[23, 24, 25, 35, 38, 40, 42]). In addition to what was previously mentioned about the effectiveness of this method in solving various non-relativistic quantum mechanics problems, its applications extended to the relativistic state. By way of examples, to draw the attention of the reader and the researcher alike, we mention its previous applications successfully on each of the deformed Klein-Gordon equation [62-79], the deformed Dirac equation [45, 80, 81, 82, 83, 84], in addition, the deformed Duffin-Kemmer equation [85, 86, 87, 88]. Thus, the reduced Schrödinger equation (without star product) can be written as:

$$H_{nc}^{esc}(\hat{p}_\mu, \hat{x}_\mu) \Psi(\vec{r}) = E_{nc}^{esc} \Psi(\vec{r}). \quad (23)$$

Which can be written in the equivalent form:

$$\frac{d^2 R_{nl}(r)}{dr^2} + 2\mu(E_{nl} - V_{esc}(\hat{r}, E_{nl})) R_{nl}(r) = 0. \quad (24)$$

In 3D-NCPS symmetries, the Hamiltonian operator $H_{nc}^{esc}(\hat{p}_i, \hat{x}_i)$ and new effective potential $V_{esc}(\hat{r}, E_{nl})$ for the improved energy-dependent screened Coulomb potential can be expressed as:

$$H_{nc}^{esc}(\hat{p}_\mu, \hat{x}_\mu) \equiv H_{esc} \left(\hat{x}_\mu = x_\mu - \frac{\theta^{\mu\nu}}{2} p_\nu, \hat{p}_\mu = p_\mu + \frac{\bar{\theta}^{\mu\nu}}{2} x_\nu \right), \quad (25)$$

and

$$V(r) \Rightarrow V_{esc}(\hat{r}, E_{nl}) = -\frac{A(1+gE_{nl}) \exp(-\alpha\hat{r})}{\hat{r}}. \quad (26)$$

After straightforward calculations, we can obtain the important terms $(-\frac{A(1+gE_{nl})}{\hat{r}})$ and $(-\frac{A(1+gE_{nl}) \exp(-\alpha\hat{r})}{\hat{r}})$ which will be used to determine the improved energy-dependent screened Coulomb potential in 3D-NCPS symmetries as:

$$-\frac{A(1+gE_{nl})}{\hat{r}} = -\frac{A(1+gE_{nl})}{r} - \frac{A(1+gE_{nl})\vec{\mathbf{L}}\vec{\Theta}}{2r^3} + O(\Theta^2), \quad (27)$$

and

$$\exp(-\alpha\hat{r}) = \exp(-\alpha r) + \frac{\alpha \exp(-\alpha r) \vec{\mathbf{L}}\vec{\boldsymbol{\theta}}}{2r} + O(\theta^2). \quad (28)$$

By combining Eqs. (27) and (28), we can easily find the expression of $(-\frac{A(1+gE_{nl})\exp(-\alpha\hat{r})}{\hat{r}})$ in 3D-NCPS symmetries:

$$-\frac{A(1+gE_{nl})\exp(-\alpha\hat{r})}{\hat{r}} = -\frac{A(1+gE_{nl})\exp(-\alpha r)}{r} - A(1+gE_{nl})\left(\frac{\exp(-\alpha r)}{2r^3} + \frac{\alpha \exp(-\alpha r)}{2r^2}\right)\vec{\mathbf{L}}\vec{\boldsymbol{\theta}} + O(\theta^2). \quad (29)$$

By making the substitution above Eq. (29) into Eq. (25), we obtain that our worked Hamiltonian operator $H_{nc}^{esc}(\hat{p}_\mu, \hat{x}_\mu)$ satisfies the equation in 3D-NCPS symmetries:

$$H_{nc}^{esc}(\hat{p}_\mu, \hat{x}_\mu) = H_{esc}(p_\mu, x_\mu) + H_{pert}^{esc}(r, \theta, \bar{\theta}), \quad (30)$$

where the operator $H_{esc}(p_\mu, x_\mu)$ is the usual Hamiltonian operator for hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}) under standard energy-dependent screened Coulomb potential in 3D-NRQM symmetries:

$$H_{esc}(p_\mu, x_\mu) = \frac{p^2}{2\mu} - \frac{A(1+gE_{nl})\exp(-\alpha r)}{r}, \quad (31)$$

while the rest three-terms are proportional with two infinitesimal parameters (θ and $\bar{\theta}$) and then we can be considered as perturbation terms $H_{pert}^{esc}(r, \theta, \bar{\theta})$ in 3D-NCPS symmetries as:

$$H_{pert}^{esc}(r, \theta, \bar{\theta}) = -\frac{A(1+gE_{nl})\exp(-\alpha r)}{2r^2}\left(\alpha + \frac{1}{r}\right)\vec{\mathbf{L}}\vec{\boldsymbol{\theta}} + \frac{\vec{\mathbf{L}}\vec{\boldsymbol{\theta}}}{2\mu}. \quad (32)$$

The newly induced part $H_{pert}^{esc}(r, \theta, \bar{\theta})$ appears as the result of the deformation of the non-commutativity phase-space symmetries. In recent work, we can disregard the second term in $H_{pert}^{esc}(r, \theta, \bar{\theta})$ the operator because we are interested in the corrections of the first order (θ and $\bar{\theta}$).

B. Perturbed spin-orbit Hamiltonian operator for hydrogen-like atoms under improved energy-dependent-screened Coulomb potential

In this subsection, we aimed to obtain the applicable physical form of the induced Hamiltonian that appears due to the non-commutativity of phase-space properties. To achieve this goal, we replace both $(\vec{\mathbf{L}}\vec{\boldsymbol{\theta}}, \vec{\mathbf{L}}\vec{\boldsymbol{\theta}})$ with useful physical forms $\beta(\bar{\theta}\vec{\mathbf{L}}\vec{\boldsymbol{\theta}}, \bar{\theta}\vec{\mathbf{L}}\vec{\boldsymbol{\theta}})$, respectively. We replace $H_{pert}^{esc}(r, \theta, \bar{\theta})$ by new useful form $H_{so}^{esc}(r, \theta, \bar{\theta})$ to obtain the new physical form of the perturbed Hamiltonian as follows [62, 63, 64, 65, 66, 67]:

$$H_{so}^{esc}(r, \theta, \bar{\theta}) \equiv \beta\left\{\frac{\bar{\theta}}{2\mu} - \frac{A(1+gE_{nl})}{2r^2}\left(\alpha + \frac{1}{r}\right)\exp(-\alpha r)\theta\right\}\vec{\mathbf{L}}\vec{\boldsymbol{\theta}} \quad (33)$$

with

$$\left.\begin{aligned} \theta^2 &= \theta_{12}^2 + \theta_{23}^2 + \theta_{13}^2 \\ \bar{\theta}^2 &= \bar{\theta}_{12}^2 + \bar{\theta}_{23}^2 + \bar{\theta}_{13}^2 \end{aligned}\right\} \quad (34)$$

And $\beta \approx \frac{1}{137}$ is the atomic fine structure constant while $\vec{\mathbf{S}}$ denoting the spin of the hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}). As a direct result, the spin-orbit interactions $H_{so}^{esc}(r, \theta, \bar{\theta})$ induced automatically due to the deformation of the phase-space. To the best of our knowledge, in ordinary QM symmetries, we can replace the quantum spin-orbit coupling $\vec{\mathbf{L}}\vec{\mathbf{S}}$ with $\frac{1}{2}(\vec{\mathbf{J}}^2 - \vec{\mathbf{L}}^2 - \vec{\mathbf{S}}^2)$, here $\vec{\mathbf{J}} = \vec{\mathbf{L}} + \vec{\mathbf{S}}$ is the global momentum of the hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}). These data allow us to formulate the following equation:

$$H_{pert}^{esc}(r, \theta, \bar{\theta}) = \frac{\beta}{2}\left\{\frac{\bar{\theta}}{2\mu} - \frac{A(1+gE_{nl})}{2r^2}\left(\alpha + \frac{1}{r}\right)\exp(-\alpha r)\theta\right\}(\vec{\mathbf{J}}^2 - \vec{\mathbf{L}}^2 - \vec{\mathbf{S}}^2). \quad (35)$$

For hydrogen-like atoms, the physical eigenvalues j of the $\vec{\mathbf{J}}$ can be confined to the interval $|l - 1/2| \leq j \leq |l + 1/2|$. We need to determine two-sided bounds to the eigenvalues $k(j, l, s)$ of the operator $(\vec{\mathbf{J}}^2 - \vec{\mathbf{L}}^2 - \vec{\mathbf{S}}^2)$ as follows:

$$k(j, l, s) = \begin{cases} k_- \left(j = l - \frac{1}{2}, l, s = \frac{1}{2}\right) & \text{for spin_down,} \\ k_+ \left(j = l + \frac{1}{2}, l, s = \frac{1}{2}\right) & \text{for spin_up.} \end{cases} \quad (36)$$

On the other hand, it is possible to determine a diagonal matrix $H_{so}^{esc}(r, \theta, \bar{\theta})$ of order (3×3) with diagonal elements $(H_{so}^{esc})_{11}$, $(H_{so}^{esc})_{22}$ and $(H_{so}^{esc})_{33} = 0$ as follows:

$$\left.\begin{aligned} (H_{so}^{esc})_{11} &= \alpha\beta k_+(l) \\ \left\{\frac{\bar{\theta}}{2\mu} - \frac{A(1+gE_{nl})}{2r^2}\left(\alpha + \frac{1}{r}\right)\exp(-\alpha r)\theta\right\} & \text{if } j = l + \frac{1}{2}, \end{aligned}\right\} \quad (37)$$

and

$$\left.\begin{aligned} (H_{so}^{esc})_{22} &= \beta k_-(l) \\ \left\{\frac{\bar{\theta}}{2\mu} - \frac{A(1+gE_{nl})}{2r^2}\left(\alpha + \frac{1}{r}\right)\exp(-\alpha r)\theta\right\} & \text{if } j = l - \frac{1}{2}. \end{aligned}\right\} \quad (38)$$

After straightforward calculation, the new radial function $R_{nl}(r)$ satisfies the following differential equation in 3D-NCPS for hydrogen-like atoms such as He^+ , Li^{2+} and Be^{3+} under improved energy-dependent screened Coulomb potential:

$$\frac{d^2 R_{nl}(r)}{dr^2} + 2\mu(E_{nc}^{esc} - V_{nc\text{-eff}}^{esc}(r, E_{nl}))R_{nl}(r) = 0, \quad (39)$$

with

$$V_{nc\text{-eff}}^{esc}(r, E_{nl}) = V(r, E_{nl}) + \frac{l(l+1)}{2\mu r^2} +$$

$$\beta \left\{ \frac{\bar{\theta}}{2\mu} - \frac{A(1+gE_{nl})}{2r^2} \left(\alpha + \frac{1}{r} \right) \exp(-ar) \theta \right\} \overrightarrow{\text{LS}}. \quad (40)$$

The above equation cannot be solved analytically for any state because of the potential centrifugal term and the studied potential itself. Therefore, in the present work, we consider the following approximation type suggested by (Greene and Aldrich) and Dong *et al.* for them [26, 89, 90]:

$$\frac{1}{r^2} \approx \frac{4\alpha^2 \exp(-2ar)}{(1-\exp(-2ar))^2}. \quad (41)$$

Allow us to obtain the following useful approximations:

$$\begin{cases} \frac{\exp(-ar)}{r^2} \approx \frac{4\alpha^2 z^{\frac{3}{2}}}{(1-z)^2}, \\ \frac{\exp(-ar)}{r^3} \approx \frac{8\alpha^3 z^2}{(1-z)^3}. \end{cases} \quad (42)$$

This allows the application of standard perturbation theory to determine the non-relativistic energy corrections E_{so}^{esc} of hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}) at the first order of two infinitesimal parameters (θ and $\bar{\theta}$) due to phase-space non-commutativity properties, instead of solving exactly the DSE for the effective potential $V_{nc\text{-eff}}^{esc}(r, E_{nl})$ given by Eq. (40). Thus, we now attempt to obtain corrected energy for the generalized effective potential $V_{nc\text{-eff}}^{esc}(r, E_{nl})$ given by the previous approximations:

$$V_{nc\text{-eff}}^{esc}(z, E_{nl}) = -\frac{2\alpha A(1+gE_{nl})z}{1-z} + \frac{4\alpha^2 l(l+1)}{2\mu(1-z)^2} + \beta \left\{ \frac{\bar{\theta}}{2\mu} - A(1+gE_{nl}) \left(\frac{4\alpha^3 z^2}{(1-z)^3} + \frac{2\alpha^2 z^{3/2}}{(1-z)^2} \right) \theta \right\} \overrightarrow{\text{LS}}. \quad (43)$$

The principal goal of this subsection is to determine the energies ($E_{so-n}^{u-esc}(k_-, n, \alpha, A, j, l, s) \equiv E_{so-n}^{u-esc}$ and $E_{so-n}^{d-esc}(k_-, n, \alpha, A, j, l, s) \equiv E_{so-n}^{d-esc}$) produced from the perturbed Hamiltonian operator $H_{so}^{esc}(r, \Theta, \bar{\theta})$ corresponding to the two polarities ($j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$) at the first order of two parameters θ and $\bar{\theta}$ for hydrogen-like atoms for (n, l) states by applying standard perturbation theory and through the structure constants which specified the dimensionality of improved energy-dependent screened Coulomb potential of hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}). Thus, we obtain the following results with the shorthand notations E_{so-n}^{u-esc} and E_{so-n}^{d-esc} of the corrected energy spectrum:

$$E_{so-n}^{u-esc} = \beta k_+ B_{nl}^2 \int_0^{+\infty} z^2 \sqrt{\varepsilon_{nl}^2} (1-z)^{2G_l^2} \left[P_n \left(2\sqrt{\varepsilon_{nl}^2} z^{2G_l-1} \right) (1-2z) \right]^2 \left\{ \frac{\bar{\theta}}{2\mu} - A(1+gE_{nl}) \left(\frac{4\alpha^3 z^2}{(1-s)^3} + \frac{2\alpha^2 z^{\frac{3}{2}}}{(1-s)^2} \right) \theta \right\} dr, \quad (44)$$

and

$$E_{so-n}^{d-esc} = \beta k_- B_{nl}^2 \int_0^{+\infty} z^2 \sqrt{\varepsilon_{nl}^2} (1-z)^{2G_l^2} \left[P_n \left(2\sqrt{\varepsilon_{nl}^2} z^{2G_l-1} \right) (1-2z) \right]^2 \left\{ \frac{\bar{\theta}}{2\mu} - A(1+gE_{nl}) \left(\frac{4\alpha^3 z^2}{(1-s)^3} + \frac{2\alpha^2 z^{\frac{3}{2}}}{(1-s)^2} \right) \theta \right\} dr, \quad (45)$$

We have $z = \exp(-2ar)$, and this allows us to obtain ($dr = -\frac{1}{2\alpha} \frac{dz}{z}$). After introducing a new variable ($s = 1-2z$), we obtain ($dr = \frac{1}{2\alpha} \frac{ds}{1-s}$) and ($1-z = \frac{s+1}{2}$). According to this change of variable, the approximations in Eq. (42) become:

$$\frac{\exp(-ar)}{r^2} \approx \frac{4\alpha^2 z^{3/2}}{(1-z)^2} = \frac{4\sqrt{2}\alpha^2(1-s)^{3/2}}{(1+s)^2}, \quad (46a)$$

and

$$\frac{\exp(-ar)}{r^3} \approx \frac{8\alpha^3 z^2}{(1-z)^3} = \frac{16\alpha^3(1-s)^2}{(1+s)^3}, \quad (46b)$$

This allows us to reformulate Eqs. (44.1) and (44.2) as follows:

$$\frac{2^{2G_l^2+2} \sqrt{\varepsilon_{nl}^2+1} \alpha E_{so-n}^{u-esc}}{B_{nl}^2} = \beta k_+ \int_{-1}^{+1} (1-s)^2 \sqrt{\varepsilon_{nl}^2-1} (1+z)^{2G_l^2} \left[P_n \left(2\sqrt{\varepsilon_{nl}^2} z^{2G_l-1} \right) (s) \right]^2 \left\{ -A(1+gE_{nl}) \left(\frac{8\alpha(1-s)^2}{(1+s)^3} + \frac{2\sqrt{2}(1-s)^{3/2}}{(1+s)^2} \right) \theta - \frac{\bar{\theta}}{2\mu} \right\} dz, \quad (47)$$

and

$$\frac{2^{2G_l^2+2} \sqrt{\varepsilon_{nl}^2+1} \alpha E_{so-n}^{d-esc}}{B_{nl}^2} = \beta k_- \int_{-1}^{+1} (1-s)^2 \sqrt{\varepsilon_{nl}^2-1} (1+z)^{2G_l^2} \left[P_n \left(2\sqrt{\varepsilon_{nl}^2} z^{2G_l-1} \right) (s) \right]^2 \left\{ -A(1+gE_{nl}) \left(\frac{8\alpha(1-s)^2}{(1+s)^3} + \frac{2\sqrt{2}(1-s)^{3/2}}{(1+s)^2} \right) \theta - \frac{\bar{\theta}}{2\mu} \right\} dz. \quad (48)$$

Which can also be rewritten in the non-relativistic energy corrections $E_{so-n}^{u-esc}(k_+, n, \alpha, A, g, j, l, s)$ and $E_{so-n}^{d-esc}(k_-, n, \alpha, A, g, j, l, s)$ at the first order of two infinitesimal parameters (Θ and $\bar{\theta}$) for the hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}) as follows:

$$E_{so-n}^{u-esc} = \beta B_{nl}^2 k_+ \left\{ \theta \sum_{i=1}^2 T_i(n, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} T_3(n, l, \alpha, g) \right\}, \quad (49)$$

and

$$E_{so-n}^{d-esc} = \beta B_{nl}^2 k_- \left\{ \Theta \sum_{i=1}^2 T_i(n, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} T_3(n, l, \alpha, g) \right\}. \quad (50)$$

Here, the 2 factors $T_i(n, l, \alpha, A)$ ($i = 1, 2$) and $T_3(n, l, \alpha, g)$ are given by:

$$T_1(n, l, \alpha, A, g) \equiv \frac{-8\alpha A(1+gE_{nl})}{2^{2G_l^2+2}\sqrt{\varepsilon_{nl}^2+1}\alpha} \int_{-1}^{+1} (1-s)^{2\sqrt{\varepsilon_{nl}^2+1}} (1+s)^{2G_l^2-3} \left[P_n \left(2\sqrt{\varepsilon_{nl}^2+1} \frac{s}{\alpha} \right) \right]^2 ds, \quad (51)$$

$$T_2(n, l, \alpha, A, g) \equiv \frac{-2\sqrt{2}A(1+gE_{nl})}{2^{2G_l^2+2}\sqrt{\varepsilon_{nl}^2+1}\alpha} \int_{-1}^{+1} (1-s)^{2\sqrt{\varepsilon_{nl}^2+1/2}} (1+s)^{2G_l^2-2} \left[P_n \left(2\sqrt{\varepsilon_{nl}^2+1} \frac{s}{\alpha} \right) \right]^2 ds, \quad (52)$$

and

$$T_3(n, l, \alpha, g) \equiv -\frac{1}{2^{2G_l^2+2}\sqrt{\varepsilon_{nl}^2+1}\alpha} \int_{-1}^{+1} (1-s)^{2\sqrt{\varepsilon_{nl}^2-1}} (1+s)^{2G_l^2} \left[P_n \left(2\sqrt{\varepsilon_{nl}^2+1} \frac{s}{\alpha} \right) \right]^2 ds. \quad (53)$$

It is natural to consider $T_3(n, l, \alpha, g)$ an equal one because the radial part of the wave function is normalized. For the ground state with a quantum number ($n = 0, l$), the Jacobi polynomial reduces to the $P_0 \left(2\sqrt{\varepsilon_{nl}^2+1} \frac{s}{\alpha} \right) = 1$. This allows us to rewrite the above three factors in Eqs. (51) and (52) to the following simple form:

$$T_1(0, l, \alpha, A, g) \equiv \frac{-8\alpha A(1+gE_{0l})}{2^{2G_l^2+2}\sqrt{\varepsilon_{0l}^2+1}\alpha} \int_{-1}^{+1} (1-s)^{2\sqrt{\varepsilon_{0l}^2+1}} (1+s)^{2G_l^2-3} ds, \quad (54)$$

and

$$T_2(0, l, \alpha, A, g) \equiv \frac{-2\sqrt{2}A(1+gE_{0l})}{2^{2G_l^2+2}\sqrt{\varepsilon_{0l}^2+1}\alpha} \int_{-1}^{+1} (1-s)^{2\sqrt{\varepsilon_{0l}^2+1/2}} (1+s)^{2G_l^2-2} ds. \quad (55)$$

where ε_{0l} and E_{0l} are given by:

$$\varepsilon_{0l}^2 = -\left(\frac{\mu E_{0l}}{2\alpha^2} - \frac{l(l+1)}{12} \right), \quad (56)$$

and

$$E_{0l} = -\frac{\alpha^2}{2\mu} \left[\left(\frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1)} \right)^2 - \frac{l(l+1)}{3} \right]. \quad (57)$$

Comparing Eqs. (52) and (53) with the integral of the form[90]:

$$\int_{-1}^{+1} (1-p)^{n+\alpha} (1+p)^{n+\beta} dp = 2^{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+1)}. \quad (58)$$

After straightforward calculations, we obtain the 2-factors as follows:

$$T_1(0, l, \alpha, A, g) \equiv \frac{-2\alpha A(1+gE_{0l})}{\alpha} \frac{\Gamma(2\sqrt{\varepsilon_{0l}^2+2})\Gamma(2G_l^2-2)}{(2\sqrt{\varepsilon_{0l}^2+2G_l^2-1})\Gamma(2\sqrt{\varepsilon_{0l}^2+2G_l^2-1})}, \quad (59)$$

and

$$T_2(0, l, \alpha, A, g) \equiv \frac{-A(1+gE_{0l})}{\alpha} \frac{\Gamma(2\sqrt{\varepsilon_{0l}^2+\frac{3}{2}})\Gamma(2G_l^2-1)}{(2\sqrt{\varepsilon_{0l}^2+2G_l^2-\frac{1}{2}})\Gamma(2\sqrt{\varepsilon_{0l}^2+2G_l^2-\frac{1}{2}})}. \quad (60)$$

Substituting Eqs. (59) and (60) in Eqs. (49) and (50), we obtain non-relativistic energy corrections for the ground state $E_{so-0}^{u-esc}(k_+, 0, \alpha, A, j, l, s)$ and $E_{so-0}^{d-esc}(k_-, 0, \alpha, A, j, l, s)$ at the first order of two infinitesimal parameters (Θ and $\bar{\theta}$) for hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}) corresponding to $j = l + 1/2$ and $j = l - 1/2$ as:

$$E_{so-0}^{u-esc} = \beta B_{0l}^2 k_+ \left\{ \Theta T(0, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} \right\}, \quad (61)$$

and

$$E_{so-0}^{d-esc} = \beta B_{0l}^2 k_- \left\{ \Theta T(0, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} \right\}, \quad (62)$$

with

$$T(0, l, \alpha, A, g) = \sum_{i=1}^2 T_i(0, l, \alpha, A, g).$$

For the first excited state ($n = 1, l$), the Jacobi polynomial $P_1^{(\alpha, \beta)}(s) = \alpha + 1 + (\alpha + \beta + 2) \frac{s-1}{2}$, which gives $P_1 \left(2\sqrt{\varepsilon_{1l}^2+1} \frac{s}{\alpha} \right) = g_{1l} + h_{1l}(1-s)$ with $g_{1l} = 2\sqrt{\varepsilon_{1l}^2+1}$, $h_{1l} = \sqrt{\varepsilon_{1l}^2+1} + G_l + 1/2$ and $\varepsilon_{1l}^2 = -\left(\frac{\mu E_{1l}}{2\alpha^2} - \frac{l(l+1)}{12} \right)$ while E_{1l} denoting the energy of the first excited state for the standard energy-dependent screened Coulomb potential:

$$E_{1l} = -\frac{\alpha^2}{2\mu} \left[\left(\frac{3}{2} + \sqrt{\frac{1}{4} + l(l+1)} - \frac{\mu A(1+gE_{nl})}{\alpha(3/2+\sqrt{1/4+l(l+1)})} \right)^2 - \frac{l(l+1)}{3} \right]. \quad (63)$$

Thus, the 2 factors in Eqs. (49) and (50) reduce to the following simple form:

$$\begin{cases} T_1(1, l, \alpha, A, g) \equiv T_{11}(1, l, \alpha, A, g) + T_{12}(1, l, \alpha, A, g) \\ + T_{13}(1, l, \alpha, A, g), \\ T_2(1, l, \alpha, A, g) \equiv T_{21}(1, l, \alpha, A, g) + T_{22}(1, l, \alpha, A, g) \\ + T_{23}(1, l, \alpha, A, g). \end{cases} \quad (64)$$

Where the six elements are given in the following form:

$$\begin{cases} T_{11} = \frac{-8\alpha A(1+gE1l)}{2^2 G_l^2 + 2\sqrt{\varepsilon_{1l}^2} \alpha} g_{1l}^2 \int_{-1}^{+1} (1-s)^2 \sqrt{\varepsilon_{1l}^2 + 1} (1+s)^{2G_l^2 - 3} ds, \\ T_{12} = \frac{-8\alpha A(1+gE1l)}{2^2 G_l^2 + 2\sqrt{\varepsilon_{1l}^2} \alpha} g_{1l}^2 \int_{-1}^{+1} (1-s)^2 \sqrt{\varepsilon_{1l}^2 + 2} (1+s)^{2G_l^2 - 3} ds, \\ T_{13} = \frac{-8\alpha A(1+gE1l)}{2^2 G_l^2 + 2\sqrt{\varepsilon_{1l}^2} \alpha} g_{1l}^2 \int_{-1}^{+1} (1-s)^2 \sqrt{\varepsilon_{1l}^2 + 3} (1+s)^{2G_l^2 - 3} ds, \end{cases} \quad (65)$$

and

$$\begin{cases} T_{21} = \frac{-2\sqrt{2}A(1+gE1l)}{2^2 G_l^2 + 2\sqrt{\varepsilon_{1l}^2} \alpha} g_{1l}^2 \int_{-1}^{+1} (1-s)^2 \sqrt{\varepsilon_{1l}^2 + \frac{1}{2}} (1+s)^{2G_l^2 - 2} ds, \\ T_{22} = \frac{-2\sqrt{2}A(1+gE1l)}{2^2 G_l^2 + 2\sqrt{\varepsilon_{1l}^2} \alpha} g_{1l}^2 \int_{-1}^{+1} (1-s)^2 \sqrt{\varepsilon_{1l}^2 + \frac{3}{2}} (1+s)^{2G_l^2 - 2} ds, \\ T_{23} = \frac{-2\sqrt{2}A(1+gE1l)}{2^2 G_l^2 + 2\sqrt{\varepsilon_{1l}^2} \alpha} g_{1l}^2 \int_{-1}^{+1} (1-s)^2 \sqrt{\varepsilon_{1l}^2 + \frac{5}{2}} (1+s)^{2G_l^2 - 2} ds. \end{cases} \quad (66)$$

We apply the integral in Eq. (56) to obtain the following results:

$$\begin{cases} T_{11} = -2A(1+gE1l) \frac{g_{1l}^2 \Gamma(2\sqrt{\varepsilon_{1l}^2 + 2}) \Gamma(2G_l^2 - 2)}{(2\sqrt{\varepsilon_{1l}^2 + 2G_l^2 - 1}) \Gamma(2\sqrt{\varepsilon_{1l}^2 + 2G_l^2 - 1})}, \\ T_{12} = -8A(1+gE1l) \frac{g_{1l}^2 \Gamma(2\sqrt{\varepsilon_{1l}^2 + 3}) \Gamma(2G_l^2 - 2)}{(2\sqrt{\varepsilon_{1l}^2 + 2G_l^2}) \Gamma(2\sqrt{\varepsilon_{1l}^2 + 2G_l^2})}, \\ T_{13} = -8\alpha A(1+gE1l) \frac{g_{1l}^2 \Gamma(2\sqrt{\varepsilon_{1l}^2 + 3 + 1}) \Gamma(2G_l^2 - 2)}{(2\sqrt{\varepsilon_{1l}^2 + 2G_l^2 + 1}) \Gamma(2\sqrt{\varepsilon_{1l}^2 + 2G_l^2 + 1})}, \end{cases} \quad (67)$$

and

$$\begin{cases} T_{21} = \frac{-A(1+gE1l)}{\alpha} \frac{g_{1l}^2 \Gamma(2\sqrt{\varepsilon_{1l}^2 + \frac{3}{2}}) \Gamma(2G_l^2 - 1)}{(2\sqrt{\varepsilon_{1l}^2 + 2G_l^2 - \frac{1}{2}}) \Gamma(2\sqrt{\varepsilon_{1l}^2 + 2G_l^2 - \frac{1}{2}})}, \\ T_{22} = \frac{-4A(1+gE1l)}{\alpha} \frac{g_{1l}^2 \Gamma(2\sqrt{\varepsilon_{1l}^2 + 5/2}) \Gamma(2G_l^2 - 1)}{(2\sqrt{\varepsilon_{1l}^2 + 2G_l^2 + 1/2}) \Gamma(2\sqrt{\varepsilon_{1l}^2 + 2G_l^2 + 1/2})}, \\ T_{23} = \frac{-2A(1+gE1l)}{\alpha} \frac{g_{1l}^2 \Gamma(2\sqrt{\varepsilon_{1l}^2 + \frac{7}{2}}) \Gamma(2G_l^2 - 1)}{(2\sqrt{\varepsilon_{1l}^2 + 2G_l^2 + \frac{1}{2}}) \Gamma(2\sqrt{\varepsilon_{1l}^2 + 2G_l^2 + \frac{1}{2}})}. \end{cases} \quad (68)$$

This allows us to obtain non-relativistic energy corrections for the first excited state $E_{so-1}^{u-esc}(k_+, 1, \alpha, A, g, j, l, s)$ and $E_{so-1}^{d-esc}(k_-, 1, \alpha, A, g, j, l, s)$ at the first order of two

infinitesimal parameters (θ and $\bar{\theta}$) for the hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}) corresponding to $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ under improved energy-dependent screened Coulomb potential, in 3D-NCPS symmetries as:

$$E_{so-1}^{u-esc} = \beta B_{1l}^2 k_+ \left\{ \theta T(1, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} \right\}, \quad (69)$$

and

$$E_{so-1}^{d-esc} = \beta B_{1l}^2 k_- \left\{ \theta T(1, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} \right\}, \quad (70)$$

with

$$T(1, l, \alpha, A, g) = \sum_{i=1}^2 T_i(1, l, \alpha, A, g).$$

Allow us to generalize this procedure to get non-relativistic energy corrections $E_{so-n}^{u-esc}(k_+, n, \alpha, A, g, j, l, s)$ and $E_{so-n}^{d-esc}(k_-, n, \alpha, A, g, j, l, s)$ for any excited state (n, l), the non-relativistic energy corrections at the first order of two infinitesimal parameters (θ and $\bar{\theta}$) for the hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}) corresponding to $j = l + \frac{1}{2}$ and $j = l - 1/2$ under improved energy depended screened Coulomb potential, in 3D-NCPS symmetries as follows:

$$E_{so-n}^{u-esc} = \beta B_{nl}^2 k_+ \left\{ \theta T(n, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} \right\}, \quad (71)$$

and

$$E_{so-n}^{d-esc} = \beta B_{nl}^2 k_- \left\{ \theta T(n, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} \right\}, \quad (72)$$

with

$$T(n, l, \alpha, A, g) = \sum_{i=1}^2 T_i(n, l, \alpha, A, g).$$

This allows us to deduce the following important physical results for hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}) under improved energy-dependent screened Coulomb potential:

$$H_{so}^{esc}(r, \theta, \bar{\theta}) \frac{R_{nl}(r)}{r} Y_l^m(\theta, \phi) = \begin{cases} E_{so-n}^{u-esc}(k_+, n, \alpha, A, g, j, l, s) \frac{R_{nl}(r)}{r} Y_l^m(\theta, \phi) \text{ for } j = l + 1/2 \\ E_{so-n}^{d-esc}(k_-, n, \alpha, A, g, j, l, s) \frac{R_{nl}(r)}{r} Y_l^m(\theta, \phi) \text{ for } j = l - 1/2 \end{cases} \quad (73)$$

C. The bound-state solution of the modified Zeeman effect for the improved energy depended on the screened Coulomb potential

In this subsection, having obtained the energy spectrum E_{so-n}^{u-esc} and E_{so-n}^{d-esc} which is self-produced from the perturbed Hamiltonian operator $H_{so}^{esc}(r, \theta, \bar{\theta})$ corresponding to ($j = l \pm 1/2$) at first order of two parameters (Θ and $\bar{\theta}$) for hydrogen-like atoms for (n, l) states. For our purposes, we

are interested in finding a new second automatically important symmetry for improved energy-dependent screened Coulomb potential. This physical phenomenon is induced automatically by the influence of a self-uniform magnetic field $\vec{\kappa}$ if we make the following two simultaneous transformations to ensure that previous calculations are not repeated:

$$(\theta, \bar{\theta}) \rightarrow (\lambda, \bar{\sigma})\kappa. \quad (74)$$

Here λ and $\bar{\sigma}$ are just two infinitesimal real proportional constants so that the following physical units are identical $[\lambda][\kappa] \equiv [\theta]$ and $[\bar{\sigma}][\kappa] \equiv [\bar{\theta}]$. We choose the magnetic field to simplify the mathematical calculations without compromising the physical content $\vec{\kappa} = \kappa e_z$. Then we transform the spin-orbit coupling to the new physical phenomena as follows:

$$\left\{ -\frac{A(1+gEnl) \exp(-ar)}{2r^2} \left(\alpha + \frac{1}{r} \right) \vec{\theta} + \frac{\bar{\theta}}{2\mu} \right\} \vec{L} \rightarrow \left\{ -\lambda \frac{A(1+gEnl) \exp(-ar)}{2r^2} \left(\alpha + \frac{1}{r} \right) + \frac{\bar{\sigma}}{2\mu} \right\} \kappa L_z. \quad (75)$$

This allowed deriving the improved magnetic Hamiltonian operator $H_z^{esc}(r, \lambda, \bar{\sigma})$ for previous hydrogen-like atoms under improved energy depended screened Coulomb potential in local 3D-NCPS symmetries as:

$$H_z^{esc}(r, \lambda, \bar{\sigma}) = \left\{ -\lambda \frac{A(1+gEnl) \exp(-ar)}{2r^2} \left(\alpha + \frac{1}{r} \right) + \frac{\bar{\sigma}}{2\mu} \right\} \kappa_{mod}^z. \quad (76)$$

Here $\kappa_z \equiv -S\kappa$ denote to the usual Zeeman effect in commutative quantum mechanics, while $\kappa_{mod}^z \equiv \vec{\kappa} J - \kappa_z$ is the improved Zeeman effect in 3D-NCPS symmetries. To obtain the exact new magnetic modifications of energy for the ground state, the first excited state and $(n, l, m)^{th}$ excited states of hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}) under improved energy depended on screened Coulomb potential $E_{m-0}^{esc}(0, \alpha, A, g, m, s)$, $E_{m-1}^{esc}(1, \alpha, A, g, m, s)$ and $E_{m-n}^{esc}(n \geq 2, \alpha, A, g, m, s)$ we just replace $(k_+(j, l, s)$ and $(\theta, \bar{\theta})$) in Eqs. (61), (69), and (71) by the following parameters $(m$ and $(\lambda, \bar{\sigma})\kappa$), respectively:

$$E_{m-1}^{esc}(0, \alpha, A, g, m, s) = \beta B_{0l}^2 \kappa \left(\lambda T(0, l, \alpha, A, g) + \frac{\bar{\sigma}}{2\mu} \right) m, \quad (77)$$

$$E_{m-1}^{esc}(1, \alpha, A, g, m, s) = \beta B_{1l}^2 \kappa \left(\lambda T(1, l, \alpha, A, g) + \frac{\bar{\sigma}}{2\mu} \right) m, \quad (78)$$

and

$$E_{m-n}^{esc}(n, \alpha, A, g, m, s) = \beta B_{nl}^2 \kappa \left(\lambda T(n, \alpha, A, l) + \frac{\bar{\sigma}}{2\mu} \right) m. \quad (79)$$

The physical values of m are confined by the interval $-l \leq m \leq +l$, which allows us to fix $(2l + 1)$ values for discrete numbers m . It should be noted that the results obtained in Eqs. (75), (76) and (77) can obtain it by direct calculation by applying the formula:

$$E_{m-n}^{esc} = \langle \Psi | H_m^{esc}(r, \lambda, \bar{\sigma}) | \Psi \rangle. \quad (80)$$

That takes the following explicit relation:

$$E_{m-n}^{esc} = B_{nl}^2 m \kappa \int_0^{+\infty} s^2 \sqrt{\epsilon_{nl}^2} (1-s)^{2G_l^2} \left[P_n \left(2\sqrt{\epsilon_{nl}^2} s^{G_l-1} \right) (1-2s) \right]^2 \left\{ -A(1+gEnl) \left(\frac{4\alpha^3 s^2}{(1-s)^3} + \frac{2\alpha^2 s^{\frac{3}{2}}}{(1-s)^2} \right) \lambda + \frac{\bar{\sigma}}{2\mu} \right\} dr. \quad (81)$$

We end this subsection by addressing the important result:

$$H_z^{esc}(r, \lambda, \bar{\sigma}) \Psi_{nlm}(r, \theta, \phi) = \beta B_{nl}^2 \kappa, \left\{ \lambda T(n, l, \alpha, A, g) + \frac{\bar{\sigma}}{2\mu} \right\} m \Psi_{nlm}(r, \theta, \phi). \quad (82)$$

D. Bound state solution of perturbed Fermi gas for improved energy depended screened Coulomb potential

For our purposes, we are interested in finding a new, third-automatic, important symmetry for improved energy-dependent screened Coulomb potential at zero temperature in 3D-NCPS symmetries. This physical phenomenon is induced automatically by the influence of a perturbed Hamiltonian operator $H_{per}^{esc}(r)$ which we have seen in Eq. (32), we discover these important physical phenomena when our studied system consists of N non-interacting is considered as a Fermi gas undergoing rotation with angular velocity $\vec{\Omega}$, if we make the following two simultaneous transformations to ensure that previous calculations are not repeated:

$$\left(\vec{\theta}, \vec{\bar{\theta}} \right) \rightarrow (\chi, \bar{\chi}) \vec{\Omega}, \quad (83)$$

here $(\chi, \bar{\chi})$ are two infinitesimal real proportional constants, and to simplify the calculations without compromising physical content, we choose $\vec{\Omega} = \Omega e_z$. Thus, we transform the spin-orbit coupling to the new physical phenomena as follows:

$$\left\{ -\frac{A(1+gEnl) \exp(-ar)}{2r^2} \left(\alpha + \frac{1}{r} \right) \vec{\theta} + \frac{\bar{\theta}}{2\mu} \right\} \vec{L} \rightarrow \left\{ -\chi \frac{A(1+gEnl) \exp(-ar)}{2r^2} \left(\alpha + \frac{1}{r} \right) + \frac{\bar{\chi}}{2\mu} \right\} \Omega L_z. \quad (84)$$

This allowed us to obtain the new modified Hamiltonian operator $H_f^{esc}(r, \chi, \bar{\chi})$ for previous hydrogen-like atoms

under improved energy-dependent screened Coulomb potential in local 3D-NCPS symmetries as:

$$H_f^{esc}(r, \chi, \bar{\chi}) = \left\{ -\chi \frac{A(1+gEnl) \exp(-ar)}{2r^2} \left(\alpha + \frac{1}{r} \right) + \frac{\bar{\chi}}{2\mu} \right\} \Omega L_z. \quad (85)$$

To obtain the exact NC Fermi gas modifications of energy for the ground state, the first excited state and $(n, l, m)^{th}$ excited states of Fermi gas under improved energy depended on screened Coulomb potential $E_{m-0}^{f-esc}(0, \alpha, A, g, m, s)$, $E_{m-1}^{f-esc}(1, \alpha, A, g, m, s)$ and $E_{m-n}^{f-esc}(n \geq 2, \alpha, A, g, m, s)$ we just replace $k_+(j, l, s)/k_-(j, l, s)$ and $(\Theta, \bar{\theta})$ in Eqs. (61), (69), and (71) by the following parameters m and $(\chi, \bar{\chi})\Omega$, respectively:

$$E_{m-0}^{f-esc} = \beta B_{0l}^2 \left(\chi T(0, l, \alpha, A, g) + \frac{\bar{\chi}}{2\mu} \right) \Omega m, \quad (86)$$

$$E_{m-1}^{f-esc} = \beta B_{1l}^2 \left(\chi T(1, l, \alpha, A, g) + \frac{\bar{\chi}}{2\mu} \right) \Omega m, \quad (87)$$

and

$$E_{m-n}^{f-esc} = \beta B_{nl}^2 \left(\chi T(n, \alpha, A, l, g) + \frac{\bar{\chi}}{2\mu} \right) \Omega m. \quad (88)$$

It should be noted that the results obtained in Eq. (88) can find it by direct calculation by applying the formula:

$$E_{m-n}^{f-esc} = \langle \Psi | H_f^{esc}(r, \chi, \bar{\chi}) | \Psi \rangle, \quad (89)$$

that takes the following explicit relation:

$$E_{m-n}^{f-esc} = B_{nl}^2 m \Omega \int_0^{+\infty} s^2 \sqrt{\varepsilon_{nl}^2} (1-s)^{2G_l^2} \left[P_n \left(\frac{2\sqrt{\varepsilon_{nl}^2} s^{2G_l-1}}{1-s} \right) (1-2s) \right]^2 \left\{ -A(1+gEnl) \left(\frac{4\alpha^3 s^2}{(1-s)^3} + \frac{2\alpha^2 s^{3/2}}{(1-s)^2} \right) \chi + \frac{\bar{\chi}}{2\mu} \right\} dr. \quad (90)$$

We end this subsection by addressing the critical result:

$$H_f^{esc}(r, \chi, \bar{\chi}) \Psi_{nlm}(r, \theta, \phi) = \beta B_{nl}^2 \Omega \left\{ \chi T(n, l, \alpha, A, g) + \frac{\bar{\chi}}{2\mu} \right\} m \Psi_{nlm}(r, \theta, \phi). \quad (91)$$

It is worth mentioning that Bencheikh *et al.* [91] studied rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gas in a two and three-dimensional space at zero temperature but in this study, the rotational term was added to the Hamiltonian operator, in contrast to our case, where this rotational term $(-\chi \frac{A(1+gEnl) \exp(-ar)}{2r^2} (\alpha +$

$\frac{1}{r}) \Omega \hat{L}_z)$ with the corresponding kinetic operator $\frac{\bar{\chi}}{2\mu} \Omega \hat{L}_z$ appears automatically due to the large symmetries resulting from the deformation of the phase space. It is important to note that perturbation theory cannot be utilized to find corrections of the second order (θ^2 and $\bar{\theta}^2$) because we have only employed corrections of the first order of infinitesimal non-commutative parameters (θ and $\bar{\theta}$).

VI. RESULTS AND DISCUSSION

In the previous subsections, we obtained the solution of the deformed Schrödinger equation for improved energy depended screened Coulomb potential, which is determined in Eq. (1) by using the generalized Bopp's shift method and standard perturbation theory. The energy eigenvalues are calculated in 3D-NCPS symmetries. The improved eigenenergies for the ground state, first excited state, and $(n, l, m)^{th}$ excited states of hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}) under improved energy depended screened Coulomb's potential:

$$E_{nc-0}^{esc}(0, k_+, k_-, \alpha, A, g, j, l, m, s),$$

$$E_{nc-1}^{esc}(1, k_+, k_-, \alpha, A, g, j, l, m, s),$$

and

$$E_{nc-n}^{esc}(n, k_+, k_-, \alpha, A, g, j, l, m, s),$$

with spin-1/2 are obtained in this paper based on our original results presented in Eqs. (61), (62), (69), (70), (71), (72), (77), (78), (79), (86), (87), and (88), in addition to the ordinary energy E_{nl} for improved energy-depended screened Coulomb potential model, which is presented in Eq. (17) take the form:

$$E_{nc-0}^{esc}(0, l, \alpha, A, g) = E_{0l} + \beta B_{0l}^2, \quad \left\{ \begin{aligned} & \left(\lambda T(0, l, \alpha, A, g) + \frac{\bar{\sigma}}{2\mu} \right) m + \left(\chi T(0, l, \alpha, A, g) + \frac{\bar{\chi}}{2\mu} \right) \Omega m \\ & + \beta \left(\theta T(0, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} \right) \begin{cases} k_+ & \text{for } j = l + 1/2 \\ k_- & \text{for } j = l - 1/2 \end{cases} \end{aligned} \right\} \quad (92)$$

and

$$E_{nc-1}^{esc}(1, l, \alpha, A, g) = E_{1l} + \beta B_{1l}^2$$

$$\left\{ \begin{aligned} & \left(\lambda T(1, l, \alpha, A, g) + \frac{\bar{\sigma}}{2\mu} \right) m + \left(\chi T(1, l, \alpha, A, g) + \frac{\bar{\chi}}{2\mu} \right) \Omega m \\ & + \beta \left(\theta T(1, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} \right) \begin{cases} k_+ & \text{for } j = l + 1/2 \\ k_- & \text{for } j = l - 1/2 \end{cases} \end{aligned} \right\} \quad (93)$$

here E_{0l} and E_{1l} are given previously by Eqs. (57) and (63). Thus, the generalized formula of energy spectra can be expressed as follows:

$$E_{nc-n}^{esc}(n, k_+, k_-, \alpha, A, g, j, l, m, s) = -\frac{\alpha^2}{2\mu} \left[\left(n + 1/2 + \sqrt{1/4 + l(l+1)} - \frac{\mu A(1+gEnl)}{\alpha(n+1/2+\sqrt{1/4+l(l+1)})} \right)^2 - \frac{l(l+1)}{3} \right] + \beta B_{nl}^2 \left(\chi T(n, l, \alpha, A, g) + \frac{\bar{\sigma}}{2\mu} \right) \aleph m + \left(\chi T(n, l, \alpha, A, g) + \frac{\bar{\chi}}{2\mu} \right) \Omega m + \left(\theta T(n, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} \right) \begin{cases} k_+ & \text{for } j = l + 1/2 \\ k_- & \text{for } j = l - 1/2 \end{cases} \quad (94)$$

Thus, the total energy $E_{nc-n}^{esc}(n, k_+, k_-, \alpha, A, g, j, l, m, s)$ for improved energy-dependent screened Coulomb potential in 3D-NCPS symmetries, is the sum of the ordinary part of the energy E_{nl} and the two corrections of the energy (E_{so-n}^{u-esc} , E_{so-n}^{d-esc}) for $j = l \pm 1/2$ and $E_{m-n}^{esc}(n, \alpha, A, g, j, m, s)$. This is one of the main objectives of our research. Finally, we end this section by introducing the important result of this work as follows:

$$\left(H_{esc}(r) + H_{so}^{esc}(r, \theta, \bar{\theta}) + H_z^{esc}(r, \lambda, \bar{\sigma}) \right) \Psi_{nlm}(r, \theta, \phi) = (E_{nl} + B_{nl}^2 \beta \aleph \left(\chi T(n, l, \alpha, A, g) + \frac{\bar{\sigma}}{2\mu} \right) m + \beta B_{0l}^2 \left(\chi T(0, l, \alpha, A, g) + \frac{\bar{\chi}}{2\mu} \right) \Omega m + B_{nl}^2 \begin{cases} k_+ \left(\theta T(n, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} \right) & \text{for } j = l + 1/2 \\ k_- \left(\theta T(n, l, \alpha, A, g) + \frac{\bar{\theta}}{2\mu} \right) & \text{for } j = l - 1/2 \end{cases}) \Psi_{nlm}(r, \theta, \phi). \quad (95)$$

This is one of the major drivers for this paper's topic. It is clear, that the obtained eigenvalues of energies are real, which allows us to consider the NC diagonal Hamiltonian $H_{nc}^{esc}(r, \theta, \bar{\theta}, \lambda, \bar{\sigma})$ as a Hermitian operator. Furthermore, and regarding the previously obtained results presented in Eqs. (31), (35), (74), and (89), the global Hamiltonian operator, at first order in $(\Theta$ and $\bar{\theta})$, with improved energy-dependent screened Coulomb potential for hydrogen-like atoms for $(n, l, m)^{th}$ states takes the form as:

$$H_{nc}^{esc}(r, \Theta, \bar{\theta}, \chi, \bar{\sigma}) = H_{esc}(p_\mu, x_\nu) + H_{so}^{esc}(r, \Theta, \bar{\theta}) + H_z^{esc}(r, \lambda, \bar{\sigma}) + H_f^{esc}(r, \chi, \bar{\chi}) \quad (96)$$

where $H_{esc}(p_\mu, x_\nu)$ present the Hamiltonian operator for standard energy depended screened Coulomb potential while $H_{so}^{esc}(r, \Theta, \bar{\theta})$, $H_z^{esc}(r, \lambda, \bar{\sigma})$ and $H_f^{esc}(r, \chi, \bar{\chi})$ are the perturbed spin-orbit Hamiltonian, modified Zeeman, and perturbed Fermi gas operators, respectively. This is the equation for Hydrogen-like atoms under the influence of improved energy-dependent screened Coulomb potential

interactions. It should be noted that this treatment considers only first-order terms in either $(\Theta$ or $\bar{\theta})$. It is evident to consider that the atomic quantum number m can take $(2l + 1)$ values and we have also two values for $j = l \pm \frac{1}{2}$ corresponding to up and down polarities, thus, every state of energy, in the usual three-dimensional space, for improved energy-dependent screened Coulomb potential will be $2(2l + 1)$ sub-state in 3D-NCPS symmetries. Thus, the total complete degeneracy of the new version of the energy level of the improved energy-dependent screened Coulomb potential is obtained as a sum of all allowed values of l . Total degeneracy is thus,

$$\sum_{l=0}^{n-1} 2(2l + 1) \equiv 2n^2 \quad (97)$$

Modified energy depended screened Coulomb's potential

The effect resulting from the topological deformations of phase-space led to the splitting of energy levels to become a doubling of their number in the framework of quantum mechanics known in the literature:

$$\sum_{l=0}^{n-1} (2l + 1) \equiv n^2 \quad (98)$$

Energy depended screened Coulomb's potential

This means that the energy levels shown with the 3D-NCPS treatment appear more detailed and more straightforward compared to similar energy levels obtained from QM treatment because the total complete degeneracy of obtained energy level becomes double in 3D-NCPS symmetries. Now, we can deduce the energy levels $E_{nc-na}^{esc}(n, k, \alpha, A, g, j, l, m, s) \equiv E_{nc-na}^{esc}$ of neutral atoms (the atoms that possess many external electrons equal to the number of protons in the nucleus) under improved energy-dependent screened Coulomb potential such as Sodium (11p and 11e), carbon, and gold (79 p and 79 e) atoms, which are known by their chemical symbols (Na, C and Au) respectively. We just replaced it $k_+(j, l, s)$ with the new value $k(j, l, s)$ which is determined by the following expression:

$$k(j, l, s) \equiv j(j + 1) + l(l + 1) - s(s + 1),$$

in Eq. (94) to ensure that previous calculations are not reputed:

$$E_{nc-na}^{esc} = -\frac{\alpha^2}{2\mu} \left[\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1)} - \frac{\mu A(1+gEnl)}{\alpha \left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1)} \right)} \right)^2 - \frac{l(l+1)}{3} \right] + \beta B_{nl}^2 \left[T(n, l, \alpha, A, g) (\lambda \aleph m + k(j, l, s) \theta) + \frac{1}{2\mu} (\bar{\sigma} \aleph m + k(j, l, s) \bar{\theta}) \right]. \quad (99)$$

V. CONCLUSIONS

From a modified version of the Schrödinger equation DSE interacting with improved energy-dependent screened Coulomb potential inspired in symmetries of non-commutative quantum mechanics, we have addressed the problem of the hydrogen-like atoms (He^+ , Li^{2+} , Be^{3+}) and neutral atoms (Na , C and Au). The main purpose of this manuscript is to apply the improved approximation scheme to the centrifugal term for any l -states using the generalized Bopp's, shift method and standard perturbation theory in 3D-NCPS symmetries to find the theoretical solution of DSE. We reached the following results:

- We have obtained the new version of the energy eigenvalues of Hydrogen-like atoms such as He^+ , Li^{2+} and Be^{3+} under improved energy-dependent screened Coulomb potential ($E_{nc-0}^{esc}(0, k_+, k_-, \alpha, A, g, j, l, m, s)$, $E_{nc-1}^{esc}(1, k_+, k_-, \alpha, A, g, j, l, m, s)$ and $E_{nc-n}^{esc}(n, k_+, k_-, \alpha, A, g, j, l, m, s)$) with spin-1/2 for the ground state, the first excited state, and $(n, l, m)^{th}$ excited states. The energy eigenvalues depend on (α, A, g) parameters and the discrete atomic quantum numbers (j, l, m, s) and two infinitesimal parameters $(\theta, \bar{\theta})$, which are induced by (position-position and phase-phase) non-commutativity properties (see Eqs. (92), (93), and (94))
- The energy eigenvalues of the bound states $E_{nc}^{esc}(n, k, \alpha, A, g, j, l, m, s)$ of neutral atoms (^{22}Na , ^{12}Ca and ^{158}Au) under improved energy-dependent screened Coulomb potential such as (^{22}Na , ^{12}Ca and ^{158}Au) atoms, with spin- s for $(n, l, m)^{th}$ excited states obtained as a function of the parameters of standard improved energy depended on the screened Coulomb potential (α, A, g) and the discrete atomic quantum numbers (j, l, m, s) (see Eq. (98)),
- The modified version of the Hamiltonian operator $H_{nc}^{esc}(r, \theta, \bar{\theta})$ in 3D-NCPS symmetries is the sum of the Hamiltonian operator of standard improved energy-dependent screened Coulomb potential $H_{esc}(r)$ and three operators, the first one is the modified spin-orbit interaction $H_{so}^{esc}(r, \theta, \bar{\theta})$ while the second is the modified Zeeman operator $H_z^{esc}(r, \lambda, \bar{\sigma})$ for the hydrogen-like atoms and neutral atoms while the third term $H_f^{esc}(r, \chi, \bar{\chi})$ corresponds to the Fermi gas at zero temperature.

We observe that the behavior of the non-relativistic physical or chemical system interacting with improved energy-dependent screened Coulomb in 3D-NCPS symmetries, remains similar to that in the relativistic case, where the spin effect is clearly shown spontaneously. Furthermore, when the two parameters $(\theta, \bar{\theta})$ are reduced to the limited values $(0, 0)$. The non-relativistic standard's results improved energy-dependent screened Coulomb potential recovered in Ref. [26]. Finally, this study has many applications in

different areas of physics and chemistry such as atomic physics (He^+ , Li^{2+} , Be^{3+}) and molecular physics (^{22}Na , ^{12}Ca , ^{158}Au) amongst others. In our view, this behavior significantly impacts the development of atomic and hydrogenic-like atom devices. The results of this research agree with those of previous comparable studies.

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REFERENCES

- [1] Sever, R., Tezan, C., Yeşiltaş Ö., Bucurgat, M., *Exact Solution of Effective Mass Schrödinger Equation for the Hulthén Potential*, Int. J. Theor. Phys. **49**, 2243-2248 (2008). <https://doi.org/10.1007/s10773-008-9656-7>.
- [2] Falaye, B. J., *Any l -state solutions of the Eckart potential via asymptotic iteration method*, Central European Journal Physics **10**, 960-965 (2012). <https://doi.org/10.2478/s11534-012-0047-6>.
- [3] Onate, C. A., Ikot, A. N., Onyeaju, M. C., Udoh, M. E., *Bound state solutions of D-dimensional Klein-Gordon equation with hyperbolic potential*, Karbala International Journal of Modern Science **3**, 1-7 (2017). <https://doi.org/10.1016/j.kijoms.2016.12.001>.
- [4] Dong, S. H., Factorization method in quantum mechanics (Vol. 150), (Springer Science & Business Media, 2007).
- [5] Okorie, U. S., Ikot, A. N., Edet, C. O., Rampho, G. J., Sever, R., Akpan, I. O., *Solutions of the Klein Gordon equation with generalized hyperbolic potential in D-dimensions*, Journal of Physics Communications **3**, 095015 (2019). <https://doi.org/10.1088/2399-6528/ab42c6>
- [6] Yukawa, H., *On the interaction of elementary particles. I.*, Proceedings of the Physico-Mathematical Society of Japan. 3rd Series **17**, 48-57 (1935). <https://doi.org/10.1143/PTPS.1.1>
- [7] Hamzavi, M., Movahedi, M., Thylwe, K. E., Rajabi, A. A., *Approximate analytical solution of the Yukawa potential with arbitrary angular momenta*, Chin. Phys. Lett. **29**, 080302 (2012). <https://doi.org/10.1088/0256-307x/29/8/080302>
- [8] Ikhdair, S. M., Sever, R., *A perturbative treatment for the energy levels of neutral atoms*, Int. J. Mod. Phys. A **21**, 6465-6476 (2006). <https://doi.org/10.1142/s0217751x06034240>
- [9] McEnnan, J., Kissel, L., Pratt, R.H., *Analytic perturbation theory for screened Coulomb potentials: Non-relativistic case*, Phys. Rev. A **13**, 532-559 (1976). <https://doi.org/10.1103/physreva.13.532>
- [10] Dutt, R., Varshni, Y.P., *Shifted large-N expansion for the energy levels of neutral atoms*, Z. Phys. D-Atoms, Molecules and Cluster **2**, 207-210 (1986). <https://doi.org/10.1007/bf01429074>

- [11] Rajabi, A. A., Hamzavi, M., *Approximate Analytical Solutions of the Perturbed Yukawa Potential with Centrifugal Barrier*, *Zeitschrift Für Naturforschung A* **68**, 454-460 (2013). <https://doi.org/10.5560/zna.2013-0023>
- [12] Vrscay, E. R., *Hydrogen atom with a Yukawa potential: Perturbation theory and continued-fractions-Padé approximants at large order*, *Phys. Rev. A* **33**, 1433-1436 (1986). <https://doi.org/10.1103/physreva.33.1433>
- [13] Ugalde, J. M., Sarasola, C., Lopez, X., *Atomic and molecular bound ground states of the Yukawa potential*, *Phys. Rev. A* **56**, 1642-1645 (1997). <https://doi.org/10.1103/physreva.56.1642>
- [14] Gazeau, J. P., Maquet, A., *Bound states in a Yukawa potential: A Sturmian group-theoretical approach*, *Phys. Rev. A* **20**, 727-739 (1979). <https://doi.org/10.1103/physreva.20.727>
- [15] Totsuji, H., *Theory of Critical Screening Radius of Energy Levels of Hydrogen-Like Atoms in Plasmas*, *Journal of the Physical Society of Japan* **31**, 584-590 (1971). <https://doi.org/10.1143/jpsj.31.584>
- [16] Mehta, C. H., Patil, S. H., *Nonperturbative approach to screened Coulomb potentials*, *Phys. Rev. A* **17**, 34-42 (1978). <https://doi.org/10.1103/physreva.17.34>
- [17] Dutt, R., Varshni, Y. P., *An analytic approximation for the energy levels of neutral atoms*, *Z Physik A* **313**, 143-145 (1983). <https://doi.org/10.1007/BF01417219>
- [18] Ikhdair, S., *Approximate κ -state solutions to the Dirac-Yukawa problem based on the spin and pseudospin symmetry*, *Open Physics* **10**(2), 361-381 (2012). <https://doi.org/10.2478/s11534-011-0121-5>
- [19] Ikot, A. N., Maghsoodi, E., Zarrinkamar, S., Salehi, N., Hassanabadi, H., *Pseudospin and spin symmetry of Dirac-generalized Yukawa problems with Coulomb-like tensor interaction via susyqm*, *Int. J. Mod. Phys. E* **22**, 1350052 (2013). <https://doi.org/10.1142/s0218301313500523>
- [20] Aydoğdu, O., Sever, R., *The Dirac-Yukawa problem in view of pseudospin symmetry*, *Phys. Scri.* **84**, 025005 (2011). <https://doi.org/10.1088/0031-8949/84/02/025005>
- [21] Hamzavi, M., Ikhdair, S. M., *Approximate Solution of the Duffin-Kemmer-Petiau Equation for a Vector Yukawa Potential with Arbitrary Total Angular Momenta*, *Few-Body Syst.* **54**, 1753-1763 (2012). <https://doi.org/10.1007/s00601-012-0487-y>
- [22] Hamzavi, M., Ikhdair, S. M., Thylwe, K. E., *Spinless particles in the field of unequal scalar-vector Yukawa potentials*, *Chinese Physics B* **22**, 040301 (2013). <https://doi.org/10.1088/1674-1056/22/4/040301>
- [23] Maireche, A., *Investigations on the Relativistic Interactions in One-Electron Atoms with Modified Yukawa Potential for Spin 1/2 Particles*, *International Frontier Science Letters* **11**, 29-44 (2017). <https://doi.org/10.18052/www.scipress.com/IFSL.11.29>
- [24] Maireche, A., *Non-relativistic treatment of hydrogen-like and neutral atoms subjected to the generalized perturbed Yukawa potential with centrifugal barrier in the symmetries of non-commutative quantum mechanics*, *International Journal of Geometric Methods in Modern Physics* **17**, 2050067 (2020). <https://doi.org/10.1142/S021988782050067X>
- [25] Maireche, A., *Analytical Expressions to Energy Eigenvalues of the Hydrogenic Atoms and the Heavy Light Mesons in the Framework of 3D-NCPS Symmetries Using the Generalized Bopp's Shift Method*, *Bulg. J. Phys.* **49**, 239-265 (2022). <https://doi.org/10.55318/bgjp.2022.49.3.239>
- [26] Okorie, U. S., Ikot, A. N., Amadi, P. O., Thompson, A., Ngiangia, A. T., Ibekwe, E. E., *Approximate solutions of the Schrödinger equation with energy-dependent screened Coulomb potential in D-dimensions*, *Eclética Química Journal* **45**, 40-56 (2020). <https://doi.org/10.26850/1678-4618eqj.v45.4.2020.p40-56>
- [27] Capozziello, S., Lambiase, G., Scarpetta, G., *Generalized uncertainty principle from quantum geometry*, *Int. J. Theor. Phys.* **39**, 15-22 (2000). <https://doi.org/10.1023/A:1003634814685>
- [28] Doplicher, S., Fredenhagen, K., Roberts, J. E., *Spacetime quantization induced by classical gravity*, *Phys. Lett. B* **331**, 39-44 (1994). [https://doi.org/10.1016/0370-2693\(94\)90940-7](https://doi.org/10.1016/0370-2693(94)90940-7)
- [29] Witten, E., *Reflections on the Fate of Spacetime*, *Physics Today* **49**, 24-30 (1996). <https://doi.org/10.1063/1.881493>
- [30] Kempf, A., Mangano, G., Mann, R. B., *Hilbert space representation of the minimal length uncertainty relation*, *Phys. Rev. D* **52**, 1108-1118 (1995). <https://doi.org/10.1103/physrevd.52.1108>
- [31] Scardigli, F., *Some heuristic semi-classical derivations of the Planck length, the Hawking effect and the Unruh effect*, *Nuov Cim B* **110**, 1029-1034 (1995). <https://doi.org/10.1007/BF02726152>
- [32] Adler, R. J., Santiago, D. I., *On gravity and the uncertainty principle*, *Mod. Phys. Lett. A* **14**, 1371-1381 (1999). <https://doi.org/10.1142/s0217732399001462>
- [33] Kanazawa, T., Lambiase, G., Vilasi, G., Yoshioka, A., *Non-commutative Schwarzschild geometry and generalized uncertainty principle*, *The European Physical Journal C* **79**, (2019). <https://doi.org/10.1140/epjc/s10052-019-6610-1>
- [34] Scardigli, F., *Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment*, *Phys. Lett. B* **452**, 39-44 (1999). [https://doi.org/10.1016/s0370-2693\(99\)00167-7](https://doi.org/10.1016/s0370-2693(99)00167-7)
- [35] Maireche, A., *The investigation of approximate solutions of Deformed Schrödinger Equations for the Hydrogenic Atoms, Heavy Quarkonium Systems ($Q\bar{Q} = b\bar{b}, c\bar{c}$) and Diatomic molecule Bound-State Problem under Improved Exponential, Generalized, Anharmonic Cornell potential Model in NCPS symmetries*, *Lat. Am. J. Phys. Educ.* **12**, 2304 (2022).
- [36] Ho, P. M., Kao, H. C., *Non-commutative Quantum Mechanics from Non-commutative Quantum Field Theory*, *Phys. Rev. Lett.* **88**, 151602 (2002). <https://doi.org/10.1103/physrevlett.88.151602>
- [37] Boumali, A., Aounallah, H., *Exact solutions of vector bosons in the presence of the Aharonov-Bohm and Coulomb potentials in the gravitational field of topological defects in non-commutative space-time*, *Rev. Mex. Fis.* **66**, 192-208 (2020). <https://doi.org/10.31349/RevMexFis.66.192>

- [38] Maireche, A., *The Impact of Deformed Space-Phase Parameters into HAs and HLM Systems with the Improved Hulthén Plus Hellmann Potentials Model in the Presence of Temperature-Dependent Confined Coulomb Potential Within the Framework of DSE*, *Rev. Mex. Fis.* **68**, 050702-1 (2022). <https://doi.org/10.31349/RevMexFis.68.050702>.
- [39] Gnatenko, P., *Parameters of non-commutativity in Lie-algebraic non-commutative space*, *Phys. Rev. D* **99**, 026009-1 (2019). <https://doi.org/10.1103/physrevd.99.026009>
- [40] Maireche, A., *A New Model for Describing Heavy-Light Mesons in the extended non-relativistic quark model under a new modified potential containing Cornell, Gaussian and inverse square terms in the symmetries of NCQM*, *To Phys. J.* **3**, 197-215 (2019).
- [41] Bertolami, O., Rosa, J. G., De Aragao, C. M. L., Castorina, P., Zappala, D., *Scaling of variables and the relation between non-commutative parameters in non-commutative quantum mechanics*, *Mod. Phys. Lett. A* **21**, 795–802 (2006). <https://doi.org/10.1142/s0217732306019840>
- [42] Maireche, A., *A theoretical investigation of non-relativistic bound state solution at finite temperature using the sum of modified Cornell plus inverse quadratic potential*, *Sri Lankan Journal of Physics* **21**, 11-35 (2020). <https://doi.org/10.4038/sljpv.20i0>.
- [43] Maireche, A., *Heavy-light mesons in the symmetries of extended non-relativistic quark model*, *Yanbu Journal of Engineering and Science* **17**, 51-63 (2019). <https://doi.org/10.53370/001c.23732>
- [44] Djemaï, E. F., Smail, H., *On Quantum Mechanics on Non-commutative Quantum Phase Space*, *Commun. Theor. Phys. (Beijing, China)* **41**, 837–844 (2004). <https://doi.org/10.1088/0253-6102/41/6/837>
- [45] Yuan, Yi, Kang, LI, Jian-Hua, W., Chi-Yi., C., *Spin-1/2 relativistic particle in a magnetic field in NC phase space*, *Chinese Physics C* **34**, 543–547 (2010). <https://doi.org/10.1088/1674-1137/34/5/005>
- [46] Bertolami, O., Leal, P., *Aspects of phase-space non-commutative quantum mechanics*, *Phys. Lett. B* **750**, 6–11 (2015). <https://doi.org/10.1016/j.physletb.2015.08.024>
- [47] Bastos, C., Bertolami, O., Dias, N.C., Prata, J. N., *Weyl–Wigner formulation of non-commutative quantum mechanics*, *Journal of Mathematical Physics* **49**, 072101 (2008). <https://doi.org/10.1063/1.2944996>
- [48] Zhang, J., *Fractional angular momentum in non-commutative spaces*. *Phys. Lett. B* **584**, 204–209 (2004). <https://doi.org/10.1016/j.physletb.2004.01.049>
- [49] Gamboa, J., Loewe, M., Rojas, J. C., *Non-commutative quantum mechanics*, *Phys. Rev. D* **64**, 067901 (2001). <https://doi.org/10.1103/PhysRevD.64.067901>
- [50] Chaichian, M., Jabbari, S., Tureanu, A., *Hydrogen Atom Spectrum and the Lamb Shift in Non-commutative QED*, *Phys. Rev. Lett.* **86**, 2716–2719 (2001). <https://doi.org/10.1103/physrevlett.86.2716>
- [51] Maireche, A., *Non-relativistic bound state solutions of the modified 2D-Killingbeck potential involving 2D-Killingbeck potential and some central terms for hydrogenic atoms and quarkonium system*, *J. Nano- Electron. Phys.* **11**, 04024 (2019). [https://doi.org/10.21272/jnep.11\(4\).04024](https://doi.org/10.21272/jnep.11(4).04024)
- [52] Maireche, A., *New Bound State Energies for Spherical Quantum Dots in Presence of a Confining Potential Model at Nano and Plank’s Scales*, *NanoWorld J.* **1**, 122-129 (2016). <https://doi.org/10.17756/nwj.2016-016>
- [53] De Andrade, M. A., Neves, C., *Non-commutative mapping from the symplectic formalism*, *Journal of Mathematical Physics* **59**, 012105 (2018). <https://doi.org/10.1063/1.4986964>
- [54] Abreu, E. M. C., Neves, C., Oliveira, W., *Non-commutativity from the symplectic point of view*, *Int. J. Mod. Phys. A* **21**, 5359-5369 (2006). <https://doi.org/10.1142/s0217751x06034094>
- [55] Abreu, E. M. C., Neto, J. A., Mendes, A. C. R., Neves, C., Oliveira, W., Marcial, M.V., *Lagrangian formulation for non-commutative nonlinear systems*, *Int. J. Mod. Phys. A* **27**, 1250053 (2012). <https://doi.org/10.1142/s0217751x12500534>
- [56] Mezincescu, L., *Star operation in quantum mechanics*, e-print arXiv: hep-th/0007046v2.
- [57] Wang, J., Li, K., *The HMW effect in non-commutative quantum mechanics*, *Journal of Physics A: Mathematical and Theoretical* **40**, 2197–2202 (2007). <https://doi.org/10.1088/1751-8113/40/9/021>
- [58] Li, K., Wang, J., *The topological AC effect on non-commutative phase space*, *The European Physical Journal C* **50**, 1007–1011 (2007). <https://doi.org/10.1140/epjc/s10052-007-0256-0>
- [59] Maireche, A., *A New Approach to the Approximate Analytic Solution of the Three-Dimensional Schrödinger Equation for Hydrogenic and Neutral Atoms in the Generalized Hellmann Potential Model*, *Ukr. J. Phys.* **65**, 987-1001 (2020). <https://doi.org/10.15407/ujpe65.11.987>
- [60] Gouba, L., *A comparative review of four formulations of non-commutative quantum mechanics*, *Int. J. Mod. Phys. A* **31**, 1630025 (2016). <https://doi.org/10.1142/s0217751x16300258>
- [61] Bopp, F., *La mécanique quantique est-elle une mécanique statistique classique particulière?*, *Ann. Inst. Henri Poincaré* **15**, 81 (1956).
- [62] Maireche, A., *Bound state solutions of Klein-Gordon and Schrödinger equations for arbitrary l-state with linear combination of Hulthén and Kratzer potential*, *Afr. Rev. Phys.* **15**, 19-31 (2020).
- [63] Maireche, A., *New Relativistic Bound States for Modified Pseudoharmonic Potential of Dirac Equation with Spin and Pseudo-Spin Symmetry in One-electron Atom*, *Afr. Rev. Phys.* **12**, 0018, 130-143 (2017).
- [64] Maireche, A., *Theoretical Investigation of the Modified Screened cosine Kratzer potential via Relativistic and Non-relativistic treatment in the NCQM symmetries*, *Lat. Am. J. Phys. Educ.* **14**, 3310-1 (2020).
- [65] Maireche, A., *The Klein–Gordon equation with modified Coulomb plus inverse-square potential in the non-commutative three-dimensional space*, *Modern Physics Letters A* **35**, 2050015 (2020). <https://doi.org/10.1142/s0217732320500157>
- [66] Darroodi, M., Mehraban, H., Hassanabadi, H., *The Klein–Gordon equation with the Kratzer potential in the non-commutative space*, *Modern Physics Letters A* **33**, 1850203 (2018). <https://doi.org/10.1142/s0217732318502036>
- [67] Maireche, A., *A model of modified Klein-Gordon*

- equation with modified scalar-vector Yukawa potential, *Afr. Rev Phys.* **15**:0001, 1-11 (2020).
- [68] Maireche, A., *Solutions of Klein-Gordon equation for the modified central complex potential in the symmetries of non-commutative quantum mechanics*, *Sri Lankan Journal of Physics* **22**, 1–19 (2021).
<http://doi.org/10.4038/sljpv22i1.8079>
- [69] Maireche, A., *Bound-state solutions of the modified Klein-Gordon and Schrödinger equations for arbitrary l-state with the modified morse potential in the symmetries of non-commutative quantum mechanics*. *Journal of Physical Studies* **25**, 1002 (2021).
<https://doi.org/10.30970/jps.25.1002>
- [70] Maireche, A., *New bound-state solutions of the deformed Klein-Gordon and Schrödinger equations for arbitrary l-state with modified equal vector and scalar Manning-Rosen plus a class of Yukawa potential in RNCQM and NRNCQM symmetries*, *Journal of Physical Studies* **25**, 4301 (2021).
<https://doi.org/10.30970/jps.25.4301>
- [71] Maireche, A., *The Relativistic and Non-relativistic Solutions for the Modified Unequal Mixture of Scalar and Time-Like Vector Cornell Potentials in the Symmetries of Non-commutative Quantum Mechanics*, *Jordan Journal of Physics* **14**, 59-70 (2021). <https://doi.org/10.47011/14.1.6>
- [72] Maireche, A., *The Investigation of Approximate Solutions of Deformed Klein–Gordon and Schrödinger Equations Under Modified More General Exponential Screened Coulomb Potential Plus Yukawa Potential in NCQM Symmetries*, *Few-Body Syst.* **62**, 66 (2021).
<https://doi.org/10.1007/s00601-021-01639-8>
- [73] Maireche, A., *A theoretical study of the modified equal scalar and vector Manning-Rosen potential within the deformed Klein-Gordon and Schrödinger in RNCQM and NRNCQM symmetries*, *Rev. Mex. Fís.* **67**, 050702 (2021).
<https://doi.org/10.31349/revmexfis.67.050702>
- [74] Maireche, A., *A new theoretical study of the deformed unequal scalar and vector Hellmann plus modified Kratzer potentials within the deformed Klein–Gordon equation in RNCQM symmetries*, *Modern Physics Letters A* **36**, 2150232 (2021).
<https://doi.org/10.1142/S0217732321502321>
- [75] Maireche, A., *A New Theoretical Investigation of the Modified Equal Scalar and Vector Manning-Rosen Plus Quadratic Yukawa Potential Within the Deformed Klein-Gordon and Schrödinger Equations Using the Improved Approximation of the Centrifugal Term and Bopp's Shift Method in RNCQM and NRNCQM Symmetries*, *Spin J.* **11**, No. 4, 2150029 (2021).
<https://doi.org/10.1142/S2010324721500296>
- [76] Maireche, A., *Heavy quarkonium systems for the deformed unequal scalar and vector Coulomb–Hulthén potential within the deformed effective mass Klein–Gordon equation using the improved approximation of the centrifugal term and Bopp's shift method in RNCQM symmetries*, *International Journal of Geometric Methods in Modern Physics* **18**, 2150214 (2021).
<https://doi.org/10.1142/S0219887821502133>
- [77] Maireche, A., *A Theoretical Model of Deformed Klein–Gordon Equation with Generalized Modified Screened Coulomb Plus Inversely Quadratic Yukawa Potential in RNCQM Symmetries*, *Few-Body Syst.* **62**, 12 (2021).
<https://doi.org/10.1007/s00601-021-01596-2>
- [78] Maireche, A., *Modified unequal mixture scalar vector Hulthén–Yukawa potentials model as a quark-antiquark interaction and neutral atoms via relativistic treatment using the improved approximation of the centrifugal term and Bopp's shift method*, *Few-Body Syst.* **61**, 30 (2020).
<https://doi.org/10.1007/s00601-020-01559-z>
- [79] Maireche A., *Relativistic symmetries of bosonic particles and antiparticles in the background of the position-dependent mass for the improved deformed Hulthén plus deformed type-hyperbolic potential in 3D-EQM symmetries*, *East Eur. J. Phys.* **4**, 200-2012 (2022).
<https://periodicals.karazin.ua/eejp/article/view/20865/19710>
- [80] Abyaneh, M. Z., Farhoudi, M. *Electron dynamics in non-commutative geometry with magnetic field and Zitterbewegung phenomenon*, *Eur. Phys. J. Plus* **136**, 863 (2021). <https://doi.org/10.1140/epjp/s13360-021-01855-5>
- [81] Maireche, A., *Approximate k-state solutions of the deformed Dirac equation in spatially dependent mass for the improved Eckart potential including the improved Yukawa tensor interaction in ERQM symmetries*, *International Journal of Geometric Methods in Modern Physics* **19**, 2250085 (2022).
<https://doi.org/10.1142/S0219887822500852>
- [82] Maireche, A., *On the interaction of an improved Schiöberg potential within the Yukawa tensor interaction under the background of deformed Dirac and Schrödinger equations*, *Indian J. Phys.* **96**, 10 (2022).
<https://doi.org/10.1007/s12648-022-02433-w>
- [83] Maireche, A., *Relativistic symmetries of the deformed Dirac equation through the improved Hulthén plus a class Yukawa potential including a Coulomb-like in deformed quantum mechanics*, *Journal of Physical Studies* **26**, 2001-1 (2022). <https://doi.org/10.30970/jps.26.2001>
- [84] Maireche, A., *New Approximate Solutions to a Spatially-Dependent Mass Dirac Equation for Modified Hylleraas Plus Eckart Potential with Improved Yukawa Potential as a Tensor in the DQM Framework*, *Few-Body Syst.* **63** (2022).
<https://doi.org/10.1007/s00601-022-01766-w>
- [85] Saidi, A., Sedra, M. B., *Spin-one (1+3)-dimensional DKP equation with modified Kratzer potential in the non-commutative space*, *Mod. Phys. Lett. A* **35**, 2050014 (2020).
<https://doi.org/10.1142/s0217732320500145>
- [86] Cuzinato, R. R., de Montigny, M., Pompeia, P. J., *Non-commutativity and non-inertial effects on a scalar field in a cosmic string space-time: II. Spin-zero Duffin–Kemmer–Petiau-like oscillator Class*, *Quantum Grav.* **39**, 075007 (2022). <https://doi.org/10.1088/1361-6382/ac51bc>
- [87] Houcine, A., Abdelmalek, B., *Solutions of the Duffin–Kemmer Equation in Non-Commutative Space of Cosmic String and Magnetic Monopole with Allowance for the Aharonov–Bohm and Coulomb Potentials*, *Phys. Part. Nuclei Lett.* **16**, 195–205 (2019).
<https://doi.org/10.1134/S1547477119030038>
- [88] Greene, R.L., Aldrich, C., *Variational wave functions for a screened Coulomb potential*, *Phys. Rev. A* **14**, 2363–2366 (1976). <https://doi.org/10.1103/physreva.14.2363>

[89] Dong, S. H., Qiang, W. C., Sun, G. H., Bezerra, V. B., *Analytical approximations to the l-wave solutions of the Schrödinger equation with the Eckart potential*. Journal of Physics A: Mathematical and Theoretical **40**, 10535–10540 (2007). <https://doi.org/10.1088/1751-8113/40/34/010>
[90] Gradshteyn, S., Ryzhik, I. M., Table of Integrals, Series and Products, 7th. ed. eds. Alan Jeffrey Daniel Zwillinger

(Elsevier, USA, 2007).

[91] Medjedel, S., Bencheikh, K., *Exact analytical results for density profile in Fourier space and elastic scattering function of a rotating harmonically confined ultra-cold Fermi gas*, Phys. Lett. A **383**, 1915-1921 (2019).
<https://doi.org/10.1016/j.physleta.2019.03.021>