

Deriving the Maxwell-Boltzmann speed distribution function



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Abstract

This paper is aimed at university students taking courses in exact sciences. Despite its importance, in introductory subjects in undergraduate courses in exact sciences, the demonstration of Maxwell-Boltzmann's speed distribution law is not performed, only its final expression is presented. In order to fill this deficiency, in this work we seek to show in detail, in a very didactic way, the demonstration of such a law. For this purpose, the kinetic theory of gases is initially introduced before demonstrating the Maxwell-Boltzmann velocity distribution law. It is also commented on the good agreement of the Maxwell-Boltzmann speed distribution law with experimental results, and its applicability limit with regard to relativistic speeds.

Keywords: Gas kinetics, Maxwell-Boltzmann distribution, Mean square velocity.

Resumo

Este artigo é destinado a estudantes universitários cursando disciplinas de ciências exatas. Apesar de sua importância, nas disciplinas introdutórias dos cursos de graduação em ciências exatas, não é realizada a demonstração da lei de distribuição de velocidades de Maxwell-Boltzmann, apenas é apresentada sua expressão final. Para suprir esta deficiência, neste trabalho procuramos mostrar detalhadamente, de forma bastante didática, a demonstração de tal lei. Para tanto, a teoria cinética dos gases é inicialmente introduzida antes da demonstração da lei de distribuição de velocidades de Maxwell-Boltzmann. Também é comentado a boa concordância da lei de distribuição de velocidades de Maxwell-Boltzmann com resultados experimentais, e seu limite de aplicabilidade no que diz respeito a velocidades relativísticas.

Palavras-chave: Cinética de gases, Distribuição de Maxwell-Boltzmann, Velocidade quadrática média.

I. INTRODUCTION

The temperature of any physical system is the result of the movement of the molecules and atoms that make up the system. These small parts of matter have a range of different velocities and the velocities of each particle constantly varies due to collisions with each other. Such relative velocity distribution specifies the fraction for each velocity range as a function of the system temperature. This distribution is named after its creators: the British physicist and mathematician James Clerk Maxwell (1831-1879) and the Austrian physicist Ludwig Eduard Boltzmann (1844-1906).

At the beginning of the second half of the 19th century, around 1859, Maxwell carried out studies on how the velocity modules of gas molecules were distributed in thermal equilibrium and in 1860 he demonstrated and published that the velocities of gas molecules are distributed according to the law of error distribution, which was formulated in 1795 by the German mathematician, physicist and astronomer Johann Carl Friedrich Gauss (1777-1855). In this law, the kinetic energy of the molecules is proportional to the absolute temperature T of the gas. Later,

in 1872, Boltzmann generalized this law, currently known as "Maxwell-Boltzmann law" [1].

The Maxwell-Boltzmann velocity distribution is, in short, a probability distribution that can be applied in various areas such as physics, engineering, biology, chemistry, etc. The process of evaporation of liquids [2], sunlight [3], thermal neutrons in nuclear reactors [4], income distribution [5], modeling of stock exchange indices [6], are all examples of phenomena in which the Maxwell-Boltzmann velocity distribution can be employed.

Despite its importance, most undergraduate courses in the area of exact sciences do not deal with the origin and demonstration of the Maxwell-Boltzmann velocity distribution law, presenting only its final equation. In order to fill this deficiency, in this work we seek to show in a very detailed and didactic way the demonstration of such a law. For this purpose, we initially introduce the kinetic theory of gases in Section 2 before starting the demonstration of the Maxwell-Boltzmann law, carried out in Section 3. Section 4 addresses the issue of experimental verification of the Maxwell-Boltzmann velocity distribution law and its validity limit. Section 5 is reserved for conclusions and final comments.

II. THE KINETIC THEORY OF GASES

The analysis of the movement of gas particles is very complex, as it involves an immense number of particles with their respective positions and linear moments. Normally this number is on the order of Avogadro's number ($\approx 6,022 \times 10^{23}$), which makes the direct calculation of individual physical quantities such as kinetic energy and speed unfeasible.

In this case, to extract relevant physical information from the system, a statistical approach is used on the individual magnitudes of the microscopic bodies that make up the system. These quantities are called "microscopic variables or microscopic quantities". These, in turn, imply measurable macroscopic effects in the form of pressure, temperature and volume – aspects involved in the equation of state of a given substance. These quantities are called "macroscopic variables or macroscopic quantities" [7].

Conceptually, the pressure of a gas is the ratio between the force exerted by its particles on the inner faces of the walls of a container and the internal surface area of the container. The temperature of the gas is a measure of the average kinetic energy of its atoms or molecules, and its volume is the volume of the container that contains it [8].

The average kinetic energy $\langle E \rangle$ takes into account the mean square value of the velocity of the gas particles, $\langle v^2 \rangle$, based on the following relationship:

$$\langle E \rangle = \frac{m\langle v^2 \rangle}{2}, \quad (1)$$

where m is the mass of its atoms or molecules.

We then seek a physical-mathematical model that relates the macroscopic parameters temperature (T), pressure (P) and volume of the gas (V) with the average kinetic energy and the average quadratic velocity of the particles that make up the gas.

Let there be a gas at a constant temperature T contained in a cubic recipient. The pressure that its particles exert on the internal walls depends on the transfer of momentum in the form of elastic collisions between the particles that make up the gas and the internal faces of the walls of the recipient that contains it. In this model, the gas is composed of point atoms or molecules and the forces arising from the interaction between the corpuscles, as well as their collisions, are negligible [9].

An isolated particle has mass m and speed \vec{v} with the directions and magnitudes of the free velocities. Particles are subject to Newton's laws of motion with total isotropic motion. Figure 1 illustrates the elastic collision of a particle (atom or molecule) with one of the walls of the recipient.

The transfer of linear momentum from the wall to the particle occurs, in the X direction, as follows

$$p_{x(\text{initial})} = mv_{x(\text{initial})}, \quad (2)$$

and

$$p_{x(\text{final})} = mv_{x(\text{final})}, \quad (3)$$

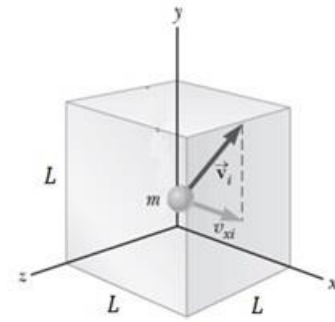


FIGURE 1. Molecule moves with speed \vec{v} towards a collision with the wall.

and considering the perfectly elastic collision we have

$$v_{x(\text{final})} = -v_{x(\text{initial})}. \quad (4)$$

Thus, the change in linear momentum¹ of the particle immediately after collision with the wall is:

$$\Delta p = p_{x(\text{final})} - p_{x(\text{initial})} = mv_{x(\text{final})} - mv_{x(\text{initial})}.$$

Using Eq. (4), this last expression becomes

$$\Delta p = -mv_{x(\text{initial})} - mv_{x(\text{initial})},$$

$$\Delta p = -2mv_{x(\text{initial})} = -2mv_x,$$

where it was defined: $v_x = v_{x(\text{initial})}$. Therefore, the momentum transferred from the gas molecule to the inner wall is

$$\Delta p = 2mv_x. \quad (5)$$

The time required for the particle to reach the front wall and return, considering a cube with edge L is:

$$v_x = \frac{2L}{\Delta t} \Rightarrow \Delta t = \frac{2L}{v_x}. \quad (6)$$

The variation of linear momentum p in relation to temporal variation consists of the physical quantity force (Newton's 2nd Law), that is:

$$\frac{\Delta p}{\Delta t} = F \Rightarrow \Delta p = F\Delta t,$$

and using Eq. (5) this last expression becomes:

$$F\Delta t = 2mv_x \Rightarrow F = \frac{2mv_x}{\Delta t}. \quad (7)$$

Substituting Eq. (6) into Eq. (7) we have:

¹ we will use p (lower case) to represent linear momentum, and P (upper case) to represent pressure.

$$F = \frac{2mv_x}{2L/v_x} \Rightarrow F = \frac{mv_x^2}{L}. \quad (8)$$

This is the expression of the force that one particle produces on the wall of the recipient. To find the pressure P exerted on this face of the wall, it is necessary to determine the force that all N particles apply to it and relate it to its area L^2

$$F_{total} = \sum_{i=1}^N \frac{mv_{xi}^2}{L},$$

$$P = \frac{F_{total}}{L^2} = \frac{1}{L^2} \sum_{i=1}^N \frac{mv_{xi}^2}{L} = \frac{m}{L^3} \sum_{i=1}^N v_{xi}^2.$$

The average of the square of the velocities in the X direction can be found by multiplying this last expression by the fraction N/N :

$$P = \left(\frac{N}{N}\right) \frac{m}{L^3} \left(\sum_{i=1}^N v_{xi}^2\right)$$

$$P = \frac{mN}{L^3} \left(\sum_{i=1}^N \frac{v_{xi}^2}{N}\right) = \frac{mN}{L^3} \langle v_x^2 \rangle$$

$$P = \frac{mN}{V} \langle v_x^2 \rangle, \quad (9)$$

where L^3 is the volume V of the cube.

The average velocities are independent of the direction adopted, as there are a very large number of particles moving randomly very quickly [10]. In this way, the modules of the average velocities are equal in any direction. Like this:

$$\langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle, \quad (10)$$

With

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle, \quad (11)$$

and then

$$\langle v^2 \rangle = 3\langle v_x^2 \rangle \Rightarrow \langle v_x^2 \rangle = \frac{1}{3} \langle v^2 \rangle. \quad (12)$$

Substituting Eq. (12) into Eq. (9) we have

$$P = \frac{mN}{V} \frac{\langle v^2 \rangle}{3} \Rightarrow PV = \frac{mN}{3} \langle v^2 \rangle. \quad (13)$$

The ideal gas equation is [11]

$$PV = nRT, \quad (14)$$

Deriving the Maxwell-Boltzmann speed distribution function

where R is the ideal gas constant and n is the number of moles. Eq. (14) can be related to Eq. (13) as follows

$$nRT = \frac{mN}{3} \langle v^2 \rangle.$$

Being $N = nN_A$, where N_A is Avogadro's number [12], the previous expression takes the form

$$nRT = \frac{mnN_A}{3} \langle v^2 \rangle \Rightarrow RT = \frac{mN_A}{3} \langle v^2 \rangle.$$

Using the molar mass M of the gas to be $M = mN_A$, the previous expression becomes

$$RT = \frac{M}{3} \langle v^2 \rangle. \quad (15)$$

Isolating $\langle v^2 \rangle$, we have

$$\langle v^2 \rangle = \frac{3RT}{M},$$

and extracting its square root, we have the square root of the mean squared value of the particle velocity, denoted by v_{rms} :

$$v_{rms} = \sqrt{\frac{3RT}{M}} \quad (16)$$

or

$$v_{rms} = \sqrt{\frac{3kT}{m}}, \quad (17)$$

where $k = R/N_A$ is the Boltzmann constant [13] (Lopes, 2010). From Eq. (15), it can be deduced that the average squared velocity of each of the components v_x , v_y and v_z is,

$$RT = \frac{M}{3} \langle v^2 \rangle = \frac{M}{3} 3\langle v_x^2 \rangle,$$

$$\langle v_x^2 \rangle = \frac{RT}{M}, \quad (18)$$

or

$$\langle v_x^2 \rangle = \frac{kT}{m}. \quad (19)$$

Soon

$$v_{x(rms)} = \sqrt{\frac{RT}{M}}, \quad (20)$$

or

$$v_{x(rms)} = \sqrt{\frac{kT}{m}}. \quad (21)$$

III. THE MAXWELL-BOLTZMANN VELOCITY DISTRIBUTION FUNCTION

The results obtained in Section 2 are very significant, since, from a macroscopic physical quantity such as temperature T , it is possible to determine the velocity v_{rms} with which the gas particles move. However, the exposed deduction leads to another hypothesis.

If there is v_{rms} , a statistical velocity parameter, there may be a distribution of probable velocities, that is, a probability density function that describes the ranges of velocities to be assumed by the particles of this gas. James Clerk Maxwell presented, in 1859, a work in which he exposed a distribution of speeds of a gas in thermal equilibrium and, in 1876, Ludwig Boltzmann arrived at the same result using a different model. Below we present a possible deduction of this speed distribution function.

Considering a three-dimensional space of molecular velocities, as illustrated in Figure 2. The objective is to find the probability of gas particles, which have different velocities, having velocity components between v_x and $v_x + dv_x$; v_y and $v_y + dv_y$; and v_z e $v_z + dv_z$.

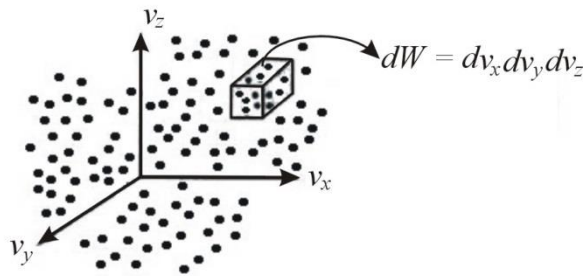


FIGURE 2. Velocities space.

Let be $f(v_x, v_y, v_z)dv_x dv_y dv_z$ a fraction of the total number of particles of a gas with components v_x , v_y and v_z comprised in the velocity range exposed in the previous paragraph. We want to find out what this speed distribution function is. As the mean square velocity depends on the gas temperature, it is assumed that this distribution also depends. Consider the gas in thermal equilibrium with a constant temperature T .

By defining the probability of continuous random variables [14], and considering that the number of molecules is large enough to adopt as continuous the values of velocities contained in an interval whose limits tend to $-\infty$ and $+\infty$ in each dimension of the velocity space, there is up

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(v_x, v_y, v_z)dv_x dv_y dv_z = 1, \quad (22)$$

or

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(v_x, v_y, v_z)dW = 1, \quad (23)$$

where dW is the volume element of the velocity space, that is $dW = dv_x dv_y dv_z$.

However, two considerations must be made, similar to those used in Section 2 to deduce the speed v_{rms} . Firstly, the components are independent, that is, the speed v_x , for example, is not linked to the values of v_y and v_z . This is true for each component in relation to the other two. Secondly, there is no preferred spatial direction in which the molecules walk, that is, space is considered isotropic.

Mathematically, these principles are expressed by adopting the same functional form for each component (spatial isotropy) and by separating the main function into autonomous functions in relation to the others (independence between components), that is

$$\int_{-\infty}^{+\infty} f(v_x)dv_x = 1, \quad (24)$$

$$\int_{-\infty}^{+\infty} f(v_y)dv_y = 1, \quad (25)$$

$$\int_{-\infty}^{+\infty} f(v_z)dv_z = 1. \quad (26)$$

As these components are independent and their functions are interpreted probabilistically, the function $f(v_x, v_y, v_z)dv_x dv_y dv_z$ is equal to the product of the component functions, that is

$$f(v_x, v_y, v_z)dv_x dv_y dv_z = f(v_x)dv_x f(v_y)dv_y f(v_z)dv_z,$$

and integrating throughout the space

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(v_x, v_y, v_z)dv_x dv_y dv_z = \\ & = \int_{-\infty}^{+\infty} f(v_x)dv_x \int_{-\infty}^{+\infty} f(v_y)dv_y \int_{-\infty}^{+\infty} f(v_z)dv_z. \quad (27) \end{aligned}$$

Spatial isotropy implies non-dependence on the direction adopted, as the probability of finding a given number of particles in a fraction of the velocity space must be the same as that of another region equidistant from the origin of that space. Therefore, it is necessary to consider no longer the speed components, but its modulus in the speed distribution function, that is:

$$f(v)dW = f(v_x)f(v_y)f(v_z)dW, \quad (28)$$

where $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$. Like this:

$$f(v) = f(v_x)f(v_y)f(v_z). \quad (29)$$

Taking the derivative of $f(v)$ with respect to v_x , we have

$$\begin{aligned} \frac{df(v)}{dv_x} &= \frac{df(v_x)}{dv_x} f(v_y)f(v_z) \\ \left(\frac{\partial f(v)}{\partial v}\right) \left(\frac{\partial v}{\partial v_x}\right) &= \frac{df(v_x)}{dv_x} f(v_y)f(v_z). \end{aligned} \quad (30)$$

As $v = (v_x^2 + v_y^2 + v_z^2)^{1/2}$, we have:

$$\begin{aligned} \frac{\partial v}{\partial v_x} &= \frac{1}{2}(v_x^2 + v_y^2 + v_z^2)^{-1/2} \cdot 2v_x \\ \frac{\partial v}{\partial v_x} &= \frac{v_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_x}{v} \end{aligned} \quad (31)$$

Furthermore:

$$\frac{\partial f(v)}{\partial v} = f'(v), \quad (32)$$

and

$$\frac{df(v_x)}{dv_x} = f'(v_x). \quad (33)$$

Inserting Eqs. (31), (32) and (33) into Eq. (30), we have

$$\frac{f'(v)}{v} v_x = f'(v_x)f(v_y)f(v_z). \quad (34)$$

Note that the ratio between Eq. (34) and Eq. (29) promotes the cancellation of the term $f(v_y)f(v_z)$ as follows

$$\begin{aligned} \frac{f'(v)v_x}{f(v)v} &= \frac{f'(v_x)}{f(v_x)} \\ \frac{f'(v)}{f(v)v} &= \frac{f'(v_x)}{f(v_x)v_x}. \end{aligned} \quad (35)$$

While the right side of the equality in Eq. (35) is a function that depends on one of the velocity components, it appears that the left side is only linked to the velocity modulus v . In turn, the velocity modulus is related to the values of v_x , v_y and v_z . So, for a given value of v_x , there are infinite combinations of v_y and v_z that make the above equality valid. Therefore, the above relationship only makes mathematical sense if it is equal to a constant value, that is:

$$\frac{f'(v)}{f(v)v} = \frac{f'(v_x)}{f(v_x)v_x} = \frac{f'(v_y)}{f(v_y)v_y} = \frac{f'(v_z)}{f(v_z)v_z} = \text{const.} \quad (36)$$

We will make this constant equal to -2α due to the greater ease of working with the negative exponent after integrating one of the equations above. Like this:

$$\frac{f'(v)}{f(v)v} = -2\alpha, \quad (37)$$

and

$$\frac{f'(v_x)}{f(v_x)v_x} = -2\alpha. \quad (38)$$

Inserting Eq. (33) in Eq. (38), we have

$$\begin{aligned} \frac{df(v_x)}{dv_x f(v_x)v_x} &= -2\alpha \\ \frac{df(v_x)}{f(v_x)} &= -2\alpha v_x dv_x. \end{aligned} \quad (39)$$

Integrating this last expression

$$\begin{aligned} \int \frac{df(v_x)}{f(v_x)} &= \int -2\alpha v_x dv_x, \\ \ln(f(v_x)) &= -\alpha v_x^2 + C, \\ f(v_x) &= e^{-\alpha v_x^2 + C}, \\ f(v_x) &= A e^{-\alpha v_x^2}, \end{aligned} \quad (40)$$

where $A = e^C$. The function in Eq. (40) is a Gaussian curve similar to the normal distribution [14]. With this relationship in hand, one can find $f(v)$, considering that the functions of the other components of the velocity space are

$$f(v_y) = A e^{-\alpha v_y^2}, \quad (41)$$

and

$$f(v_z) = A e^{-\alpha v_z^2}. \quad (42)$$

Inserting Eqs. (40), (41) and (42) in Eq. (29) we have

$$\begin{aligned} f(v) &= A e^{-\alpha v_x^2} A e^{-\alpha v_y^2} A e^{-\alpha v_z^2}, \\ f(v) &= A^3 e^{-\alpha(v_x^2 + v_y^2 + v_z^2)}, \\ f(v) &= A^3 e^{-\alpha v^2}. \end{aligned} \quad (43)$$

From Eq. (24) and Eq. (19) it is possible to determine the constants A and α . Thus, inserting Eq. (40) into Eq. (24) we have:

$$I_1 = \int_{-\infty}^{+\infty} A e^{-\alpha v_x^2} dv_x = 1, \quad (44)$$

and with an analogous procedure in relation to the y component

$$I_2 = \int_{-\infty}^{+\infty} A e^{-\alpha v_y^2} dv_y = 1. \quad (45)$$

The product between Eq. (44) and Eq. (45) results in

$$I_1 \cdot I_2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A^2 e^{-\alpha(v_x^2+v_y^2)} dv_y dv_x = 1. \quad (46)$$

It is observed from the symmetry of the integrals that: $I_1 = I_2 = I$. Taking a Cartesian plane with abscissa v_x and ordinate v_y , the following relations to convert such coordinates into polar ones can be used [15]:

$$v_x^2 + v_y^2 = r^2, \quad (47a)$$

$$v_x = r \cos \theta, \quad (47b)$$

$$v_y = r \sin \theta, \quad (47c)$$

$$dv_y dv_x = r dr d\theta. \quad (47d)$$

The integration limits $-\infty < v_x < +\infty$ and $-\infty < v_y < +\infty$ of Cartesian coordinates are equivalent to the limits $0 \leq r < \infty$ and $0 \leq \theta \leq 2\pi$ of polar coordinates. Thus, with the definitions presented in Eqs. (47a-d), Eq. (46) takes the form:

$$I^2 = I_1 \cdot I_2 = \int_0^{2\pi} \int_0^{+\infty} A^2 e^{-\alpha r^2} r dr d\theta = 1,$$

$$I^2 = A^2 \int_0^{2\pi} d\theta \int_0^{+\infty} e^{-\alpha r^2} r dr = 1,$$

$$I^2 = A^2 \cdot 2\pi \cdot \frac{1}{2\alpha} = A^2 \frac{\pi}{\alpha} = 1$$

$$I = A \sqrt{\frac{\pi}{\alpha}} = 1 \Rightarrow A \sqrt{\frac{\pi}{\alpha}} = 1$$

$$A = \sqrt{\frac{\alpha}{\pi}}. \quad (48)$$

To determine the constant α , the expression $\langle v_x^2 \rangle$ obtained in Section 2 is used. Conceptually, the quadratic mean of a continuous distribution, such as $\langle v_x^2 \rangle$, is given by the integral [14]

$$\langle v_x^2 \rangle = \int_{-\infty}^{+\infty} v_x^2 f(v_x) dv_x,$$

and using the expression of $\langle v_x^2 \rangle$ given by Eq. (19), the previous equation becomes

$$\frac{kT}{m} = \int_{-\infty}^{+\infty} v_x^2 f(v_x) dv_x,$$

$$\int_{-\infty}^{+\infty} v_x^2 e^{-\alpha v_x^2} dv_x = \frac{kT}{Am}.$$

Replacing the value of A given by Eq. (48) and solving the integral [16], this last expression becomes:

$$\frac{1}{2} \sqrt{\frac{\pi}{\alpha^3}} = \frac{kT}{\sqrt{\alpha/\pi m}}.$$

Isolating α

$$\alpha = \frac{m}{2kT}. \quad (49)$$

Introducing Eq. (49) into Eq. (48) we have

$$A = \sqrt{\frac{m}{2\pi kT}}. \quad (50)$$

Introducing Eqs. (49) and (50) into Eq. (43), we have

$$f(v) = \left(\sqrt{\frac{m}{2\pi kT}} \right)^3 e^{-\frac{mv^2}{2kT}}. \quad (51)$$

The product of $f(v)$ with volume element dW makes it possible to generalize such a model. With this, the function becomes dependent only on the velocity modulus, that is:

$$f(v)dW = f(v)dv_x dv_y dv_z,$$

$$f(v)dW = f(v)v^2 d\Omega dv \quad (52)$$

Note that:

$$\int_0^{4\pi} f(v)v^2 d\Omega dv = 4\pi v^2 f(v) dv. \quad (53)$$

We therefore have the Maxwell-Boltzmann velocity distribution function, designated by f_{MB} :

$$f_{MB}(v) = 4\pi \left(\sqrt{\frac{m}{2\pi kT}} \right)^3 v^2 e^{-\frac{mv^2}{2kT}}. \quad (54)$$

The graph of this distribution resembles a Gaussian with beginning at the origin, quadratic growth and exponential decrease, as illustrated in Figure 3 for two temperature values. The value adopted for the mass m was the mass of the electron. It is noted that the effect of increasing temperature makes the probability curve flatter and longer.

The interpretation, at this point, is which fraction of gas particles have a velocity of magnitude between v and $v + dv$, regardless of the direction. It also represents the probability of a particle having a speed between v and $v + dv$. Therefore, we no longer think of a cube of dimensions dv_x , dv_y and dv_z that contains it, but of a spherical shell of thickness dv in the velocity space. Figure 4 illustrates this situation.

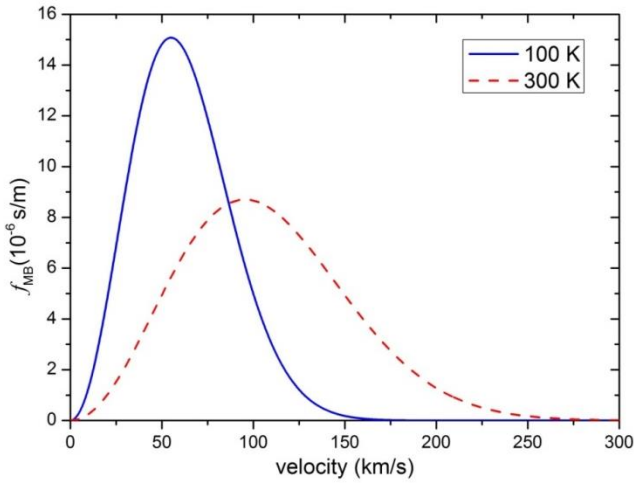


FIGURE 3. Maxwell-Boltzmann velocity distribution function for two temperature values.

The most likely speed, v_p , is the value that maximizes the function $f_{MB}(v)$. Graphically, it would be the velocity value responsible for the peak of the velocity distribution function.

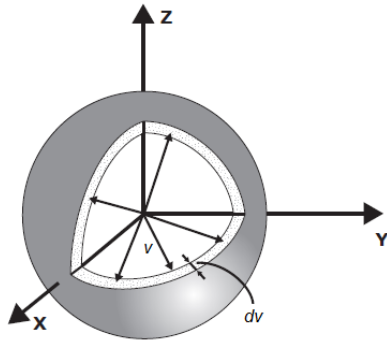


FIGURE 4. Representation of velocities space with magnitude distribution.

Therefore, taking the derivative of the Maxwell-Boltzmann distribution and equating it to zero, we have:

$$\frac{\partial(f_{MB}(v))}{\partial v} = \frac{\partial\left(4\pi\left(\sqrt{\frac{m}{2\pi kT}}\right)^3 v^2 e^{-\frac{mv^2}{2kT}}\right)}{\partial v} = 0,$$

$$2\sqrt{\frac{2}{\pi}}\left(\sqrt{\frac{m}{kT}}\right)^3 v_p e^{-\frac{m(v_p)^2}{2kT}} - \sqrt{\frac{2}{\pi}}\left(\sqrt{\frac{m}{kT}}\right)^3 \frac{m(v_p)^3}{kT} e^{-\frac{m(v_p)^2}{2kT}} = 0,$$

$$2\sqrt{\frac{2}{\pi}}\left(\sqrt{\frac{m}{kT}}\right)^3 v_p e^{-\frac{m(v_p)^2}{2kT}} = \sqrt{\frac{2}{\pi}}\left(\sqrt{\frac{m}{kT}}\right)^3 \frac{m(v_p)^3}{kT} e^{-\frac{m(v_p)^2}{2kT}},$$

and after some simplifications

$$2 = \frac{m(v_p)^2}{kT},$$

and finally

$$v_p = \sqrt{\frac{2kT}{m}}. \tag{55}$$

The average speed, $\langle v \rangle$, that is, the expected speed of this distribution, is given by the integral

$$\langle v \rangle = \int_0^{+\infty} v f_{MB}(v) dv,$$

$$\langle v \rangle = \int_0^{+\infty} v 4\pi \left(\sqrt{\frac{m}{2\pi kT}}\right)^3 v^2 e^{-\frac{mv^2}{2kT}} dv,$$

$$\langle v \rangle = 4\pi \left(\sqrt{\frac{m}{2\pi kT}}\right)^3 \int_0^{+\infty} v^3 e^{-\frac{mv^2}{2kT}} dv,$$

and solving the integral [16], we have

$$\langle v \rangle = 4\pi \left(\sqrt{\frac{m}{2\pi kT}}\right)^3 \frac{4(kT)^2}{2m^2},$$

$$\langle v \rangle = \sqrt{\frac{8kT}{\pi m}}. \tag{56}$$

The velocity v_{rms} , obtained in Section 2, can be calculated from the Maxwell-Boltzmann velocity distribution as follows:

$$\langle v^2 \rangle = \int_0^{+\infty} v^2 f_{MB}(v) dv,$$

$$\langle v^2 \rangle = \int_0^{+\infty} v^2 4\pi \left(\sqrt{\frac{m}{2\pi kT}}\right)^3 v^2 e^{-\frac{mv^2}{2kT}} dv,$$

$$\langle v^2 \rangle = 4\pi \left(\sqrt{\frac{m}{2\pi kT}}\right)^3 \int_0^{+\infty} v^4 e^{-\frac{mv^2}{2kT}} dv,$$

and solving the integral [16], we have

$$\langle v^2 \rangle = 4\pi \left(\sqrt{\frac{m}{2\pi kT}}\right)^3 \frac{3\sqrt{\pi}(2kT)^{\frac{5}{2}}}{8\sqrt{m^5}},$$

$$\langle v^2 \rangle = \frac{3kT}{m},$$

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3kT}{m}}. \tag{57}$$

It is observed that Eq. (57) is equal to Eq. (17), which highlights the validity of this distribution. In summary, the following were obtained:

- most likely velocity, v_p :

$$v_p = \sqrt{\frac{2kT}{m}},$$

- average velocity, $\langle v \rangle$:

$$\langle v \rangle = \sqrt{\frac{8kT}{\pi m}},$$

- velocity v_{rms} :

$$v_{\text{rms}} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3kT}{m}}.$$

Note that: $v_p < \langle v \rangle < v_{\text{rms}}$. Figure 5 shows the Maxwell-Boltzmann speed distribution, Eq. (54) for a temperature of 300 K, highlighting the speeds: v_p , $\langle v \rangle$ and v_{rms} . The value adopted for the mass m was the mass of the electron.

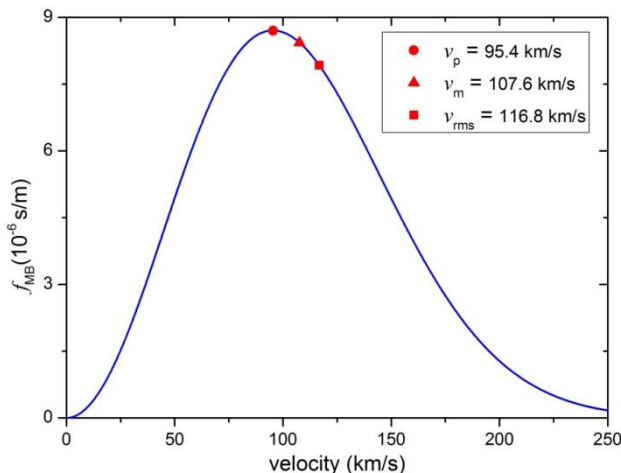


Figure 5. Maxwell-Boltzmann velocity distribution, Eq. (54), for a temperature of 300 K, highlighting the velocities: v_p , $\langle v \rangle = v_m$ and v_{rms} .

IV. AGREEMENT OF THE MAXWELL-BOLTZMANN LAW WITH EXPERIMENTAL RESULTS AND ITS APPLICABILITY LIMIT

One way to experimentally verify the validity of the Maxwell-Boltzmann velocity distribution law is to analyze how some molecular processes vary, such as the speed of chemical reactions when the temperature varies [17]. The experimental values obtained in these processes are in excellent agreement with the Maxwell-Boltzmann theory.

A more direct verification of the Maxwell-Boltzmann velocity distribution law consists of counting the number of

molecules in each velocity or energy interval. This can be done experimentally with a method that uses a mechanical speed selector composed of discs and slots that rotate with a determined angular velocity that selects the desired speeds, for example, in a recipient with an orifice containing a gas at a temperature T [18]. The experimental results obtained again confirm the Maxwell-Boltzmann predictions.

Neutrons produced in fission processes in a nuclear reactor are moderated using a material, such as water or graphite, until they reach thermal equilibrium at the temperature of the moderator. Neutrons in thermal equilibrium behave like an ideal gas and their energy distribution agrees with the Maxwell-Boltzmann velocity distribution law [19], that is, thermal neutrons follow Maxwell-Boltzmann statistics. This fact is essential in the design of nuclear reactors.

A fundamental issue to be considered is the fact that the Maxwell-Boltzmann distribution admits a non-zero probability of finding particles with speeds greater than the speed of light in a vacuum c . However, it is known from the theory of relativity that only speeds smaller than c have physical meaning. A velocity distribution function capable of dealing with this problem is the distribution function proposed by Jüttner called the Relativistic Maxwell-Jüttner Velocity Distribution Function [20, 21].

V. FINAL COMMENTS

Despite its importance, in the introductory subjects of most undergraduate courses in exact sciences, the demonstration of the Maxwell-Boltzmann velocity distribution law is not demonstrated, only its final equation is presented. In order to fill this deficiency, in this work we present a didactic and very detailed path on the construction of the Maxwell-Boltzmann velocity distribution. To this end, we begin with an introduction to the kinetic theory of gases and later demonstrate the Maxwell-Boltzmann speed distribution law. Based on this law, the most probable velocity, the average velocity and the square root of the mean squared value of the particle velocity were obtained. We also comment on the good agreement of the Maxwell-Boltzmann speed distribution law with experimental results and its applicability limit with regard to relativistic velocities.

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