The Monty Hall Problem, Information and Entropy Simulation

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Abstract
The present paper, suited for beginner college students, teaches what entropy is by examining the classic Monty Hall Problem, where a contestant is supposed to choose one of three closed doors to win a prize. We used Python simulation to calculate the probability of winning a prize by keeping or switching the initially chosen door and making an analogy to the behavior of the particles, with an unequal probability of being on both sides of a gas chamber. Boltzmann's and Shannon's entropies were introduced and their interpretation as a measure of missing information was suggested for introductory physics courses**.**

Keywords: Entropy, Monty Hall Problem, Information.

Resumen
El presente artículo, adecuado para estudiantes universitarios principiantes, enseña qué es la entropía examinando el clásico problema de Monty Hall, donde se supone que un concursante debe elegir una de tres puertas cerradas para ganar un premio. Utilizamos la simulación de Python para calcular la probabilidad de ganar un premio manteniendo o cambiando la puerta elegida inicialmente y haciendo una analogía con el comportamiento de las partículas, con una probabilidad desigual de estar a ambos lados de una cámara de gas. Se introdujeron las entropías de Boltzmann y Shannon y se sugirió su interpretación como una medida de información faltante para los cursos de introducción a la física.

Palabras clave: Entropía, Problema de Monty Hall, Información.

I. INTRODUCTION

The concept and the word entropy are quite new, being created in the 19th century. The concept of entropy and the Second Law of thermodynamics are useful tools to put into operation physical phenomena. However, entropy is an abstract concept, and it can be tough for beginning students to understand its connection to their experiences in everyday life [1, 2, 3, 4]. This is largely acknowledged in the literature, which describes the difficulties and subtleties of many facets of teaching entropy [5].

In thermal physics, the Second Law of thermodynamics, i.e., entropy increase is a quantification of the spread of energy, from spatially localized to dispersed. In statistical physics, entropy increase is related to the change in systems from a few to more accessible microstates. Another concept of entropy is Shannon's missing information [4], which is a broad, powerful, and abstract concept. Its interpretation has shed some light on the meaning of entropy. Thus, the main purpose of this paper is to suggest educators to replace the concept of entropy with the concept of missing information still in introductory physics courses. To do so, in the next section, we review the classic Monty Hall Problem [6], where a player is supposed to choose one among three closed doors to win a prize. Then, based on the premise that computer simulation helps to reinforce physics concepts [5], we use Python simulation to calculate the probability of winning a prize by keeping or switching the initially chosen door. Finally, by making an analogy to the behavior of particles, Boltzmann's and Shannon's entropies are introduced and their interpretation as a measure of missing information is suggested for introductory physics courses.

II. THE MONTY HALL PROBLEM

The Monty Hall (MH) Problem is settled on a popular TV show, "Let's Make a Deal!" in the United States of America [6]. In short, the contestant or player is supposed to choose one among three closed doors, A, B, and C. There is a prize behind one door, as opposed to the other two doors in which nothing interesting is behind them. The contestant can choose one door (let us call it door A) and take what is behind it. However, the contestant is not aware of which door hides the prize. Subsequently, the contestant has chosen door A, and the host (MH), opens another door, let us call it door B, with nothing in the background. The contestants then are supposed to choose between keeping their initial selected door, i.e., door A, or changing to the remaining door, door C. What would be the best contestant's choice to win the prize? To keep or to change their initial choice?

The unreflected claim for most of us is to stand, i.e., to keep the door initially chosen. It is intuitive, but wrong, to think that the chance that the prize is hidden behind the first chosen door is the same as it is hidden in the remaining door (50 % in both cases). One can use the frequentist approach to solve this problem. One can easily show (see Table I) that this common-sense perception is naive. As shown in Table I, there are nine equally possible exclusive scenarios. From Table I one sees that keeping the initially chosen door only gives us a 1/3 probability of being victorious. On the other hand, changing the door gives us a 2/3 probability of winning. Thus, the contestant should switch the door.

TABLE I. The nine equitably probable scenarios. The contestant is supposed to choose one among three closed doors, A, B, or C. The digits "1" and "0" stand for "winning" and "losing" respectively (see text for details).

Door initially chosen	**Door which the car is behind**	**Door opened by Monty Hall**	**Switch door**	**Keep door**
A	A	B or C	0	1
A	B	C	1	0
A	C	B	1	0
B	A	C	1	0
B	B	A or C	0	1
B	C	A	1	0
C	A	B	1	0
C	B	A	1	0
C	C	A or B	0	1
Sum			6	3
probability			6/9 =2/3	3/9=1/3

Alternatively, one can apply conditional probabilities. Bayes' Principle [6] provides the same result, although not so intuitive. It is a handy tool for probability calculations. Let *H* be a hypothesis one wishes to judge, *D* for a group of data, and *I* for the prior information one has besides the data.

$$P(H \mid DI) = P(DI \mid H)\frac{P(H \mid I)}{P(D \mid I)}. \qquad (1)$$

The prior probability $P(H|I)$ of *H* will be updated to the posterior probability $P(H|DI)$ because of getting data *D*.

We define event *A*, event *B*, and event *C*, as the events whither the prize is hidden behind door *A*, door *B*, and door *C*, respectively. It is straightforward to figure out that *P*(event *A*) =*P*(event *B*)=*P*(event *C*)=1/3. Let us also define open *A*, open *B*, and open *C*, the events in which the host opens door *A*, *B*, or *C*, after the players have made their initial choice, respectively. Let us suppose that the player chooses door *A*. Since the host is not allowed to open either door *A* (the player's initial guess) or the door which hides the prize, it is straightforward to write that *P*(open *A*)=*P*(open *B*| event *B*) = *P*(open *C*| event *C*) =0. There are two remaining doors, and each has a 50% probability of being opened. Thus, *P*(door *B*| door *A*)=0.5, where, *P*(door *B*| door *A*) is the conditional probability of event *B*, given *A*. Let us consider the two options: i) the player's initial guess is wrong. Thus, the host's option to pick a door is conditioned to if the prize is hidden behind door *B* or door *C*. Therefore, the host is supposed to choose either door *B* or door *C*. Thus, *P*(open *B*| event *C*)=*P*(open *C*| event *B*) =1. In the second case: ii) The player's initial guess is right. In that case, the host can select between the two remaining doors, *B* or *C*, at will. Thus, we can immediately write *P*(open *B* | event *A*) + *P*(open *C*| event *A*) = 1. Let us calculate the probability for the host to open door *B* using the Law of total probability:

$$P(\text{open } B) = P(\text{open } B|\text{ event } A)P(\text{event } A) + P(\text{open } B|\text{event } B)\, P(\text{event } B) + P(\text{open } B|\text{ event } C)\, P(\text{event } C). \qquad (2)$$

In the assumption that the contestant initially chooses door *A* and then the host opens door *B*, the conditional probability that the prize is behind door *A* given that the host opens door *B*, i.e. the probability of winning without changing the initial choice is

$$P(eventA \mid openB) = P(openB \mid eventA)\frac{P(openA)}{P(openB)} = \\ = \frac{1}{2} \times \frac{1/3}{1/3 \times 1/2 + 1/3 \times 0 + 1/3 \times 1} = \frac{1}{3} \qquad (3)$$

Which agrees with the frequentist's solution.

III. SIMULATION WITH PYTHON

We now describe a Python simulation to calculate the contestant's probabilities for winning a prize by keeping or switching the initially chosen door (see Supplementary Information). The program was developed using Python language by Initially importing *randint* from *random* library and the library *matplot.lib* [7]. It created three empty lists, one to store the number of victories in one game, the second one to store the number of times the number of victories repeated, and the third one to plot the graph. As executed, the program chooses randomly two values between 1 and 3, that symbolize the three doors (one which represents the contestant's chosen door, the second one which Monty opens with the goat, and the third one with the prize). The objective is to calculate the probability of success trading the door. If the values selected are equal, it will be considered a defeat, if they are different, the program will count and score a victory. This process will be repeated several times that can be adjusted and stored in the list *win*. All this process can be repeated several times that can be adjusted too, storing the new results in the list *wincount*.

Figure 1 shows the distributions of the number of wins or losses after 100 events. The simulation was repeated 10^6 times. The distribution on the left (red) shows the number of times the contestant wins after keeping the initially chosen door, while the distribution on the right (blue) shows the corresponding number of wins after switching the initially chosen door. Figure 1 shows that the mean number of wins is 66.66 out of 100 tries, corresponding to a probability of 2/3, in accordance with both the frequentist and conditional approaches discussed in the previous sections. Figure 1 also shows the standard deviations (σ = 4.71) for both distributions. One can easily see that both distributions peaked quite sharply about the mean number of wins (33.34 out of 100 in the case of keeping the initially chosen door and 66.66 out of 100 in the case of switching the initially chosen door). Figure 1 also suggests that both distributions are binomial (either keeping or switching the door), with unequal probabilities of winning, 1/3 and 2/3, respectively.

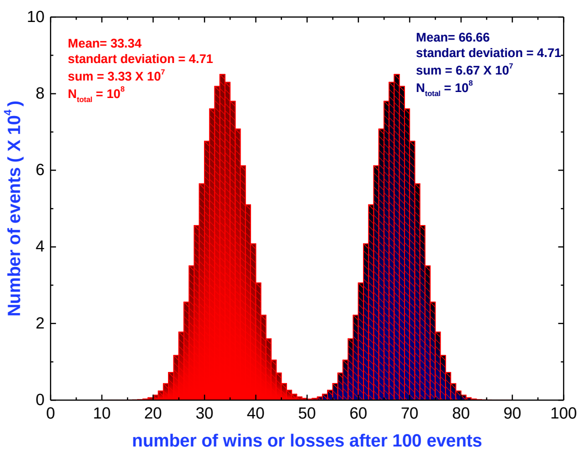

FIGURE 1. Distribution for events in which the contestants change their initial choice. The simulation consists of an ensemble of 10^6 systems, with 100 events each.

In the next two sections, one introduces Boltzmann's and Shannon's entropy, respectively. An analogy between the Monty Hall Problem and a binary model system, the distribution of molecules in a box, is proposed. We assume the system is composed of 100 independent molecules in a box. The ensemble is composed of a large number (10^6 in the present case) of identically prepared systems.

IV. BOLTZMANN'S ENTROPY AND THE MONTY HALL PROBLEM: AN ANALOGY

In the late XIX century, Boltzmann proposed a new definition of entropy, S in terms of the number of accessible microstates, Ω, constituting a particular macrostate [8]. Boltzmann's definition is the modern statistical interpretation of entropy. If there is a set of particles distributed among the Ω microstates, then Boltzmann's entropy is

$$S = k_B \ln \Omega . \tag{4}$$

The Second Law of thermodynamics was formulated by Clausius in terms of dQ/T. Thus, it was straightforward to define the entropy in J/K in the International System of Units. Consequently, Boltzmann's entropy was defined as with the same units as Clausius' entropy. However, there is no impediment to defining temperature, τ, in units of energy, i.e., $\tau \equiv k_B T$ [9]. Then, the entropy would be a dimensionless quantity. In fact, some textbooks define temperature as energy and entropy as a dimensionless quantity, see for instance [10]. A choice of base 2 in the logarithm in Eq. 2, instead of base e=2.71.., would be convenient to the interpretation of Boltzmann's entropy as missing information (next section). Thus, we can define the modified Boltzmann's entropy as (measured in bits)

$$\sigma \equiv \log_2 \Omega . \tag{5}$$

The fundamental assumption of statistical physics [10 p. 29] is that an isolated system, with no interaction with the rest of the universe, is equally probable to be in any of the accessible states, Ω, i.e., $p=\Omega^{-1}$. Although the MH problem does not constitute a thermal system (see discussion at the end of this section), within the frame of reference of information theory (see next section), one can though make an analogy between particles in boxes with the contestant's initial choice among the three likely probable doors, i.e., p=1/3, maximizing Boltzmann's modified entropy, Eq. 5, as σ=$\log_2 3 \approx 1.58$ bit.

In the case of binary systems (states 1 or 2), the binomial distribution can be used to calculate the number of microstates. One can obtain the binomial coefficients from the generating function:

$$(p_1 + p_2)^N = \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p_1^{n_1} p_2^{N-n_1} = 1 \tag{6}$$

where, in the case of single particle probabilities, $p_1 + p_2 = 1$. The probability of n_1 particles in a state 1 and $n_2 = N - n_1$ in a state 2, where $N = n_1 + n_2$, is the total number of particles in the system, given by

$$P_N(n_1) = \frac{N!}{n_1!(N-n_1)!} p_1^{n_1} p_2^{N-n_1} \tag{7}$$

$$\langle n_1 \rangle = N p_1$$
$$\langle n_2 \rangle = N p_2 \tag{8}$$

Where p_1 and p_2 are the individual probabilities of a single molecule being found in sides one of the two sides. The number of states in each configuration is

$$\Omega(N, n_1) = \frac{N!}{n_1!(N-n_1)!} \tag{9}$$

Variance σ^2 and standard deviation, σ, are defined, respectively, as

$$\sigma \equiv \sqrt{\langle n_1^2 \rangle - \langle n_1 \rangle^2}$$
$$\sigma = \sqrt{N p_1 p_2} \tag{10}$$

Since the nature of the two states systems is irrelevant to the result, i.e., it does not matter if the system is a chamber with molecules, coins, or a binary alloy, or a number of trials, one can apply the binomial distribution to the Monty Hall problem, with $p_1 = 2/3$ and $p_2 = 1/3$, where p_1 and p_2 are the winning probabilities for switching and keeping the initially chosen door, respectively. In fact, for a system of N=100 molecules (or trials), one obtains: $\langle n_1 \rangle = 66.66$ and $\langle n_2 \rangle = 33.33$, σ=4.71, as shown in Fig. 1.

The De Moivre-Laplace theorem provides a limiting value to the binomial distribution as the normal (Gaussian) distribution:

$$Lim_{N>>1} \frac{N!}{n_1!(N-n_1!)} p_1^{n_1} p_2^{N-n_1} = \\ = \frac{1}{\sqrt{2\pi N p_1 p_2}} e^{-\frac{(n_1 - p_1 N)^2}{2Np_1 p_2}} \tag{11}$$

Thus, one can appoint probabilities to maximize entropy, subject to the constraints, which are based on our information about the system. In the case of the MH problem, one can maximize Boltzmann's entropy subject to the conditional probabilities, $p_1 = 2/3$ and $p_2 = 1/3$. Thus, in the Monty Hall Problem, Boltzmann's entropy divided by Boltzmann's constant is ($100! \approx 9.33 \times 10^{+157}$; $33! \approx 8.68 \times 10^{+36}$; $66! \approx 5.44 \times 10^{+92}$, $50! \approx 3.04 \times 10^{+64}$)

$$\sigma = \log_2 \frac{100!}{33!66!} < \log_2 \frac{100!}{50!50!}, \tag{12}$$

which is smaller than a 50% probability ($p_1 = p_2 = 0.5$). The reason for this smaller value can be understood in terms of information (next section).

However, the Monty Hall Problem, or any collection of ordinary macro-objects, such as coin flipping, does not establish a thermodynamic system, as opposed as a group of molecules. The comparison between the Monty Hall Problem with particles in a box is just an analogy! The reason is that coin flipping of the number of trials in a Monty Hall problem does not exchange energy (dQ) as are molecules in a chamber. Clausius' original definition of entropy, i.e., dS = dQ/T, in a reversible process, applies to a thermodynamic system plus its surroundings.

The attribution of the thermodynamic entropy upsurge with a change in the spatial configuration of a set of macro-objects is a common misconception in physics and chemistry textbooks [3, 4]. Notwithstanding, we understand that the present analogy is an important pedagogical tool to aid the student's understanding of entropy. There is a vast literature about the misconceptions about entropy, see for instance [3, 4]. Although entropy increase can occasionally be understood in terms of an increase in disorder, it can invariably be explained in terms of the dispersal of energy, as well as misinformation, as shown in the next section.

V. SHANNON'S ENTROPY AND THE MONTY HALL

Information theory was made known in 1948 by Claude Shannon [11] to solve the problem of information transmission in communication lines. Shannon wanted to find a method for sending information in a noisy channel with minimum errors. Today, it is useful in statistical mechanics, linguistics, economics, and many other areas. The amount of information, Q_i, is determined by the probability distribution:

$$Q_i = -\log_2 p_i > 0 . \tag{13}$$

The amount of information associated with a given outcome with probability p_i (Eq. 13) decreases as p_i increases. Whenever this outcome is quite unprobable (p_i very small) the amount of information associated with obtaining that outcome is large. Thus, entropy and information, as defined in Eq. 13, are related quantities. In fact, entropy can be interpreted as a measure of uncertainty. When one knows that the system is in a particular macrostate, the entropy measures the degree of uncertainty about the specific microstate it is. Shannon Entropy is defined as the average amount of information contained in each event:

$$S = \langle Q \rangle = -\sum_i p_i \log_2 p_i \tag{14}$$

If p_i is small, the corresponding outcome is quite unprobable. Thus, it does not contribute significantly to the average information (Eq. 14). On the other hand, whenever the outcome is highly probable ($p_i \sim 1$), it contributes significantly to the average information, although carries little information content (Eq. 13).

It also gives us a measure of our uncertainty about a system, based on our limited knowledge of its properties. Information is a physics quantity. Then, the initial entropy associated with formation about where the prize is:

$$S_o = -\sum_{i=1}^{3} \frac{1}{3} \log_2 \frac{1}{3} = \log_2 3 \cong 1.58 \text{ bit}, \tag{15}$$

which agrees with the modified Boltzmann's entropy Eq. 5. The Shannon entropy for a two-outcome (event *A* or event *B*) random variable p ($0 < p < 1$), is given by

$$S(p) = -p \log_2 p - (1-p) \log_2 (1-p) . \tag{16}$$

The behavior of $S(p)$ in Eq. 15 as a function of p is plotted in Fig. 2. $S(p)$ reaches a maximum when $p = 1/2$, the numerical value of the missing information in this case is one. This is also called one bit (binary digit) of information. From Fig. 2, one sees that $S(p)$ is a minimum ($S(p)$=0) when $p = 0$ or 1. If $p = 1$, one knows for sure that event A takes place. If $p = 0$, one knows for sure that event B takes place. In both cases the missing information is null. The behaviors of both outcomes with probabilities $p_1 = p$, and $p_2 = 1 - p$, simply mirrors images of each other.

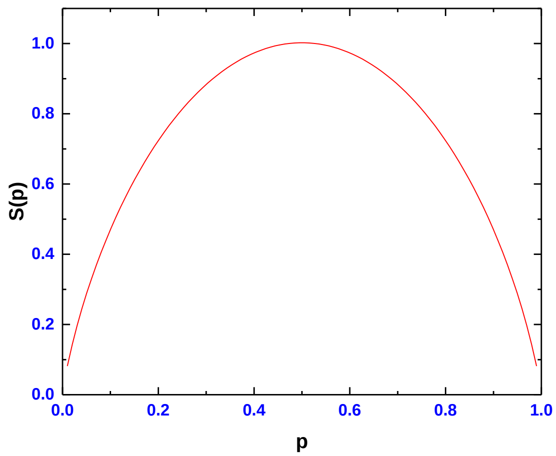

FIGURE 2. The function $S(p)$ (Eq. 16) for two outcomes.

Going back to the MH Problem, let us assume that the contestant's initial choice is door *A*. As discussed above, the new entropy after the player has made the initial choice is:

$$S = \frac{1}{3}\log_2 3 + \frac{2}{3}\log_2 \frac{3}{2} \cong 0.93\,\text{bit}, \tag{17}$$

one knows that, in this case, it is more likely to win the prize if one changes our initial choice (p=2/3). The player now has more information than in the case where the two events (changing or keeping) are equally probable. We can use this 0.07 bit of information to bet on changing our initial choice, and on average, we can win 66 % of the time.

The larger the number of doors, the greater the amount of information one would need to locate the prize. For instance, for N equiprobable doors (p=1/N), Shannon's entropy would be $S = \sum_{i=1}^{N} \frac{1}{N}\log_2 N = \log_2 N$. In other words, the larger the number of doors, the larger the amount of missing information, and the larger entropy. Once the door with the prize behind it is found, one would get all the information we need, and the entropy would be null. This is analogous to one molecule in a chamber of volume V. The chamber could be divided into N small cells each with volume V_o=V/N. Shannon's entropy, then, would be a measure of the number of binary questions one needs to inquire to localize the molecule.

Shannon's entropy (Eq. 14), apart from a constant, is equivalent to Gibbs' entropy, $S = -k_B \sum_i p_i \ln p_i$ [12], which can be demonstrated to reduce to Boltzmann's entropy [13]. The similarities among Shannon's, Gibbs', and Boltzmann's entropies provide a clue that thermodynamic entropy is a measure of one's unpredictability about a system. This uncertainty rests on one's lack of knowledge about the properties of the system, as well as about whichever of its microstates the system is in. In this frame of reference, even beginning students can figure out entropy from an information theory framework, where both Gibb's and Boltzmann's entropies are measures of the missing information required to determine the microstate of a system.

VI. CONCLUSIONS

In this paper, we suggest teaching the concept of entropy as missing information for beginning college students. For that, we contextualize the MH problem, which is quite famous, interesting, challenging, and suitable for college students. It is a simple question with a counterintuitive answer. It uses concepts of information to make a probabilistic decision.

By redefining entropy as a dimensionless quantity and the temperature in units of energy, the identification between Boltzmann's and Shannon's entropies and consequently the interpretation of entropy as a measure of missing information to specify the microstate of a system would become straightforward.

By making an analogy between particles in boxes with the contestant's initial choice among the three likely probable doors, i.e., p=1/3, maximizing Boltzmann's modified entropy, as σ=$\log_2 3 \approx 1.58$ bits, which is analog to a closed system. After the contestant makes his/her initial choice and MH opens one door, the probability is no more constant, being 1/3 for keeping the initial choice and 2/3 for changing the door. Now the analogy is with an interacting system. In that case, MH plays the role of the external interaction, and the entropy falls to 0.93 bit, which means that the contestant now has more information on the system.

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APPENDIX

Program in Python for the Monty Hall Problem (comments in Portuguese)

```
from random import randint
win = 0 #win será o número de vitórias trocando de porta. Ele mudará a cada novo ciclo
wincount = [] #armazenará todos os valores de win
for n in range(0, 100): #Controla quantas vezes o ciclo será repetido (Atual de 100 ciclos)
  for c in range(0, 100): #Controla quantos jogos terão em cada ciclo (Atual de 100 jogos)
    dc = randint(1, 3) #(door chosen) A porta escolhida pelo participante será aleatoriamente uma das três.
    rd = randint(1, 3) #(right door) A porta que contém o prêmio será aleatoriamente uma das três.
    if dc != rd:
      win = win + 1 #Caso sejam portas diferentes, obrigatoriamente ele ganharia trocando, assim uma vitória será armazenada
  wincount.append(win) #A quantidade total de vitórias neste ciclo será armazenada
  win = 0 #A quantidade de vitórias é zerada para que um novo ciclo tenha início
print(wincount) #Por fim é revelado quantas vitórias foram obtidas em cada um dos ciclos e os dados podem ser utilizados para a confecção de um gráfico.
```