A complete relativistic and nonrelativistic study of the improved trigonometric scarf potential model within the generalized tensor interaction on noncommutative space

ISSN 1870-9095

Abdelmadjid Maireche
Laboratory of Physics and Material Chemistry, Physics department, Sciences Faculty, University of M'sila -BP 239 Chebilia M'sila- Algeria.

(Received 23 January 2023, accepted 19 March 2023)

E-mail: abdelmadjid.maireche@univ-msila.dz

Abstract

Relativistic and nonrelativistic quantum mechanics formulated in a noncommutative space have recently become the object of renewed interest. We approximately solve the deformed Dirac equation for a new suggested improved Trigonometric scarf potential within the Coulomb-Hulthén-like tensor interaction (ITSP-ICHLTi) in the context of three-dimensional extended relativistic quantum mechanics symmetries with arbitrary spin-orbit coupling quantum number k . In the framework of the spin and pseudospin (p-spin) symmetry, we obtain the global new energy eigenvalue which equals the energy eigenvalue in the usual relativistic QM as the main part plus three corrected parts produced from the effect of the spin-orbit interaction, the new modified Zeeman and the rotational Fermi term, by using the parametric of well-known Bopp's shift method, standard perturbation theory, and Greene-Aldrich approximation. The new values that we get appeared sensitive to the quantum numbers ($j, k, l, m, l^p, m^p, s, s^p$), the mixed potential depths (V_0, H_c, H_H), the range of the potential α, and noncommutativity parameters (Θ, η, χ). We recovered the problems of the nonrelativistic limit of spin symmetry in the context of extended nonrelativistic quantum mechanics symmetries.

Keywords: Dirac equation, Trigonometric scarf potential, Noncommutative space, Bopp's shift method and star products.

Resumen

La mecánica cuántica relativista y no relativista formulada en un espacio no conmutativo se ha convertido recientemente en objeto de renovado interés. Resolvemos aproximadamente la ecuación de Dirac deformada para un nuevo potencial de bufanda trigonométrico mejorado sugerido dentro de la interacción tensor tipo Coulomb-Hulthén (ITSP-ICHLTi) en el contexto de simetrías tridimensionales de mecánica cuántica relativista extendida con un número cuántico k de acoplamiento de órbita-espín arbitrario. En el marco de la simetría de espín y pseudoespín (p-spin), obtenemos el nuevo valor propio de energía global que es igual al valor propio de energía en la MC relativista habitual como la parte principal más tres partes corregidas producidas por el efecto de la interacción espín-órbita, el nuevo Zeeman modificado y el término rotacional de Fermi, utilizando los parámetros paramétricos del conocido método de desplazamiento de Bopp, la teoría de la perturbación estándar y la aproximación de Greene-Aldrich. Los nuevos valores que obtenemos parecen sensibles a los números cuánticos ($j, k, l, m, l^p, m^p, s, s^p$), las profundidades potenciales mixtas (V_0, H_c, H_H), el rango del potencial α, y los parámetros de no conmutatividad (Θ, η, χ). Recuperamos los problemas del límite no relativista de la simetría de espín en el contexto de simetrías extendidas de la mecánica cuántica no relativista.

Palabras clave: Ecuación de Dirac, potencial bufanda trigonométrico, espacio no conmutativo, método de desplazamiento de Bopp y productos estrella.

I. INTRODUCTION

The end of the twentieth century was a happy event for all of humanity, as it witnessed the emergence of two great revolutions in the field of science that made humanity confront a new phase of an exciting turn toward the enlightening horizon and heralded an expected renaissance. These two enlightening events shed light on each of the macroscopic states of the universe, represented by Einstein's special and general relativity, as well as the microscopic states associated with the state of the atomic and subatomic systems. The latter is framed in quantum physical systems where the dimensions are measured on the Planck scale and the Nanoscales. These very accurate systems are governed by four fundamental equations and are well known to researchers and specialists the Schrödinger equation (SE), the Duffin-Kemmer Petiau equation (DKPE), the Klein-Gordon equation (KGE) and the Dirac equation (DE). The first equation (SE), describes the relative position at the level of low energies, while the other three equations describe the relative state at the level of high energies according to the values of spin. The relativistic effect must be considered when a particle is in a strong potential field, leading to a relativistic quantum mechanical description of such a particle, with this condition, KGE, DE, or DKPE should be used to describe a particle in a strong potential field. Furthermore, the DE's spin and pseudospin symmetries, as well as their contributions to nuclear and Hadron physics, are of major physical importance. The trigonometric scarf potential is among the important models that have been used in this context. Wei *et al.* (2010) used a straightforward algebraic approach to investigate the exact solution to DE with scalar and vector trigonometric scarf potentials under the condition of spin symmetry and presented the transcendental energy equation and the spinor waveforms [1]. It can be used to construct a periodic potential in solid-state physics and is also used to describe one-dimensional crystal models [2, 3]. L'evai *et al.* investigated the real and complex energy spectra of the Trigonometric scarf potential within the framework of the PT-symmetric quantum mechanics [4]. Quesne also used exceptional orthogonal polynomials to expand the Trigonometric scarf potential as the extended Trigonometric scarf potential, finding that the extended version had the same energy spectrum as the usual trigonometric scarf potential with new wave functions [5]. Within the framework of an approximation scheme to the centrifugal barrier, Falaye and Oyewumi obtained solutions of the Dirac equation with spin and pseudo-spin symmetry for the scalar and vector trigonometric scarf potentials in dimensions D, as well as the energy spectrum and two-component spinor eigenfunctions [6]. Onate *et al.* investigated the solutions of spin and pseudospin symmetries under the effect of the Trigonometric scarf potential in the presence of a new tensor interaction and obtained the nonrelativistic equation by taking the nonrelativistic limit of the spin symmetry [7]. The main goal of this research is to use Bopp's shift method [8, 9, 10, 11] to investigate the spin symmetry and pseudospin symmetry features of Trigonometric scarf potentials within the context of Dirac theory but within the framework of new symmetries that are more comprehensive than the symmetries of quantum mechanics known in the literature, resulting in the deformation of space-space. Another recent area of research that has received a lot of attention is the study of physical and chemical systems in a new phase-space known as noncommutative quantum mechanics (NCQM), which is a more generalized version of the usual quantum mechanics. NCQM symmetries are based on the novel postulates $\left[x_{nc\mu}^{(s,h,i)^*}, x_{nc\nu}^{(s,h,i)}\right] \neq 0$ and $\left[p_{nc\mu}^{(s,h,i)^*}, p_{nc\nu}^{(s,h,i)}\right] \neq 0$, which form noncumulative space-space (NCSS) and noncumulative phase-phase (NCPP), respectively, as well as the conventional quantum mechanics postulates which generated the form $\left[x_{nc\mu}^{(s,h,i)^*}, p_{nc\nu}^{(s,h,i)}\right] \neq 0$. The researchers believe that this expanded framework provides hope for solving many of the problems observed in quantum gravity, string theory, the standard model's divergence problem, quantum field theory regularization schemes and the study of low energy effective theories of D-branes in background magnetic fields [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. NCSS and NCPP are important tools for improving the current features of various quantum systems. whereas Connes [23] introduced the geometric analysis of NCSS and NCPP in 1991 and 1994 [24, 25]. With a nonzero B-field, Seiberg and Witten extend previous ideas concerning the advent of NC geometry in string theory and derive a new form of gauge fields in noncommutative gauge theory [26]. Among the potential goals of NCSS and NCPP is that the emergence of new quantum fluctuations can cancel the observed unwanted divergences or the infinities that appear to cause short-range in-field theories that include gravitational theory [27]. Through this new study, we dig deeper into the study of this potential to look at the possibility of other applications at the nano level. The research reported in the present paper was motivated by the fact that the study of the improved Trigonometric scarf potential, including a generalized (Coulomb-Hulthén)-like tensor interaction (ITSP-ICHLTi) in the DDT symmetries, has not been reported in the available literature. In this work, the vector and scalar ITSP-ICHLTi model ($V_{ts}(r_{nc}), S_{ts}(r_{nc})$) to be employed is defined as:

$$V_{ts}(r_{nc}) = V_{ts}(r) - \frac{1}{2r}\frac{\partial V_{ts}(r)}{\partial r}\begin{cases} \mathbf{L}\Theta + O(\Theta^2) = V_{ts}^{s}(r), \\ \text{for spin-sy.} \\ \mathbf{L}^{p}\Theta + O(\Theta^2) = V_{ts}^{p}(r), \\ \text{for p-spin-sy.} \end{cases} \tag{1}$$

and

$$S_{ts}(r_{nc}) = S_{ts}(r) - \frac{1}{2r}\frac{\partial S_{ts}(r)}{\partial r}\begin{cases} \mathbf{L}\Theta + O(\Theta^2) = S_{ts}^{s}(r), \\ \text{for spin-sy.} \\ \mathbf{L}^{p}\Theta + O(\Theta^2) = S_{ts}^{p}(r), \\ \text{for p-spin-sy.} \end{cases} \tag{2}$$

where $(V_{ts}(r), S_{ts}(r))$ are the vector and scalar potentials according to the view of RQM known in the literature [6, 7]:

$$\begin{cases} V_{ts}(r) = -\frac{V_0}{\sin^2(\alpha r)}, \\ S_{ts}(r) = -\frac{S_0}{\sin^2(\alpha r)}, \end{cases} \tag{3}$$

where V_0/S_0 are the potential depths, α is the screening parameter, $(r_{nc} \text{ and } r)$ are the distance between the two particles in the deformation of Dirac theory symmetries and QM symmetries, respectively. The two couplings ($\mathbf{L\Theta}$ and $\mathbf{L}^p\Theta$) are the scalar product of the usual components of the angular momentum operators ($\mathbf{L}(L_x, L_y, L_z)$ / $\mathbf{L}^p(L_x^p, L_y, L_z^p)$) and Θ is the modified noncommutativity vector $\Theta(\theta_{12}, \theta_{23}, \theta_{13})/2$ which presents the noncommutativity elements parameter. In the case of G_{NC}, the noncentral generators can be suitably realized as self-adjoint differential operators ($x_{nc\mu}^{(s,h,i)}$, $p_{nc\nu}^{(s,h,i)}$) appear in three varieties the first one is the canonical structure (CS), the second is the Lie structure (LS), while the last corresponds to the quantum plane (QP) in the representations of Schrödinger, Heisenberg, and interactions pictures, obeying the following set of commutation relations [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]:

$$\begin{cases} \left[x_{nc\mu}^{(s,h,i)^*}, p_{nc\nu}^{(s,h,i)} \right] = i\hbar_{eff}\delta_{\mu\nu}, \\ \left[p_{nc\mu}^{(s,h,i)^*}, p_{nc\nu}^{(s,h,i)} \right] = 0, \end{cases} \tag{4a}$$

and

$$\left[x_{nc\mu}^{(s,h,i)^*}, \Delta x_{nc\nu}^{(s,h,i)} \right] = \begin{cases} i\theta_{\mu\nu} : \theta_{\mu\nu} \in IC \text{ For CS,} \\ if_{\mu\nu}^{\alpha}\Delta x_{nc\alpha}^{(s,h,i)} : f_{\mu\nu}^{\alpha} \in IC \text{ For LS,} \\ iC_{\mu\nu}^{\alpha\delta}\Delta x_{nc\alpha}^{(s,h,i)}\Delta x_{nc\delta}^{(s,h,i)} : C_{\mu\nu}^{\alpha\delta} \in IC \text{ For QP.} \end{cases} \tag{4b}$$

which will lead to the following simplified relations within the framework of quantum mechanics, known in the literature as,

$$\begin{cases} \left[x_{\mu}^{(s,h,i)}, p_{\nu}^{(s,h,i)} \right] = i\hbar\delta_{\mu\nu}, \\ \left[x_{\mu}^{(s,h,i)}, x_{\nu}^{(s,h,i)} \right] = 0. \end{cases} \tag{4c}$$

We have used natural units $\hbar = c = 1$. Thus, in the present investigation, we define the non-commutativity of quantum theory, in which coordinates $\left\{x_{nc\mu}^{(s,h,i)}\right\}$ are non-commuted and momenta $\left\{p_{nc\mu}^{(s,h,i)}\right\}$ are commuted. Here $x_{nc\mu}^{(s,h,i)} = \left(x_{nc\mu}^{s}, x_{nc\mu}^{h}, x_{ncnc\mu}^{i}\right)$ and $p_{nc\mu}^{(s,h,i)} = \left(p_{nc\mu}^{s}, p_{nc\mu}^{h}, p_{nc\mu}^{i}\right)$ are the generalized coordinates and corresponding generalizing coordinates in the DDT symmetries, respectively, and IC is the complex number field. In the RQM symmetries, $x_{\mu}^{(s,h,i)} = \left(x_{\mu}^{s}, x_{\mu}^{h}, x_{\mu}^{i}\right)$ and $p_{\mu}^{(s,h,i)} = \left(p_{\mu}^{s}, p_{\mu}^{h}, p_{\mu}^{i}\right)$ are corresponding coordinates. Furthermore, the usual uncertainty relation corresponding to Eq. (4c) will be extended to two uncertainties in the new form symmetry as follows:

$$\left|\Delta x_{nc\mu}^{(s,h,i)}\Delta p_{nc\nu}^{(s,h,i)}\right| \geq \hbar_{eff}\delta_{\mu\nu}/2 \;, \tag{5}$$

and

$$\left|\Delta x_{nc\mu}^{(s,h,i)}\Delta x_{nc\nu}^{(s,h,i)}\right| \;\geq\; \begin{cases} \left|\theta_{\mu\nu}\right|/2 \text{ For CS,} \\ L_{\mu\nu}/2 \text{ For LS,} \\ K_{\mu\nu}/2 \text{ For QP.} \end{cases} \tag{6}$$

here $L_{\mu\nu}$ and $K_{\mu\nu}$ are present the following average values

$$\left|\left\langle \sum_{\alpha}^{3}\left(f_{\mu\nu}^{\alpha}x_{\alpha}^{(s,h,i)}\right)\right\rangle\right| \text{ and } \left|\left\langle \sum_{\alpha,\beta}^{3}\left(C_{\mu\nu}^{\alpha\beta}x_{\alpha}^{(s,h,i)}x_{\beta}^{(s,h,i)}\right)\right\rangle\right| \text{, respectively.}$$

The incertitude relation in Eq. (5) is obtained as a result of the generalization of the second part of Eq. (4b) to the first part of Eq. (4a), while the second uncertainty relation in Eq. (6) is the result of the deformation of space-space that appears from the second part of Eq. (4a) that is divided into three varieties. It is important to note that Eqs. (4a) are covariant equations (the same behavior $x_{nc\mu}^{(s,h,i)}$) under the Lorentz transformation, which includes boosts and/or rotations of the observer's inertial frame. We have extended the modified equal time noncommutative canonical commutation relations (METNCCCRs) to include the Heisenberg and interaction pictures in DDT. Here $\hbar_{eff} \cong \hbar$ [39] is the effective Planck constant, $\theta_{\mu\nu} = \varepsilon_{\mu\nu}\theta$ (θ is the noncommutative parameter, and $\varepsilon_{\mu\nu}$ is simply an antisymmetric number, $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} = 1$ with and $\varepsilon_{\varepsilon\varepsilon} = 0$) which is an infinitesimal parameter if compared to the energy values and elements of antisymmetric 3×3 real matrices, and $\delta_{\mu\nu}$ is the Kronecker symbol. The symbol $*$ represents the Weyl-Moyal star product, which is generalized between two ordinary functions $h(x)g(x)$ to the new deformed form $h(x)*g(x)$ in the symmetries of deformation of Dirac theory, known as the star-product determined by [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51]:

$$h(x)*g(x) = \begin{cases} \exp\left(i\varepsilon^{\mu\nu}\theta\partial_{\mu}^{x}\partial_{\nu}^{x}\right)(hg)(x) \text{ For CS,} \\ \exp\left(\frac{i}{2}x_{nc\mu}^{(s,h,i)}g_{k}\left(i\partial_{\mu}^{x}, i\partial_{\nu}^{x}\right)\right)(hg)(x) \text{ For LS,} \\ iq^{G\left(u,v,\partial_{\mu}^{u},\partial_{\nu}^{v}\right)}h(u,v)g(u',v')\Big|_{\substack{v'\to v \\ u'\to u}} \text{ For QP.} \end{cases}$$

here $g_{\alpha}(k,p)$ equal the value:

$$g_{\alpha}(k,p) = -k_{\mu}p_{\nu}f_{k}^{\nu\nu} + \frac{1}{6}k_{\mu}p_{\nu}\left(p_{\alpha}-k_{\alpha}\right)f_{l}^{\nu\nu}f_{m}^{l\alpha} + ...$$

In the current paper, we apply the MASCCCRs in the DDT, which allows us to rewrite $(h*g)(x)$ to the following simple form (at the first order of the noncommutativity parameter $\varepsilon^{\mu\nu}\theta$) as follows [52, 53, 54, 55, 56, 57]

$$\begin{aligned} (h*g)(x) &= \exp\left(i\varepsilon^{\mu\nu}\theta\partial_{\mu}^{x}\partial_{\nu}^{x}\right)(hg)(x), \\ &\approx (hg)(x) - \frac{i\varepsilon^{\mu\nu}\theta}{2}\partial_{\mu}^{x}h\partial_{\nu}^{x}g\Big|_{x^{\mu}=x^{\nu}} + O\left(\theta^{2}\right). \end{aligned} \tag{7}$$

Possible values for indices (μ,ν) are (1,2,3) and $O(\theta^2)$ stand for the second and higher-order terms of the NC parameter θ. Physically, the second term in Eq. (7) presents the effects of space-space noncommutativity. The outline for our paper is the following: The first section includes the scope and purpose of our investigation, while the remaining parts of the paper are structured as follows: A review of the DE with the Trigonometric scarf potential, including a generalized (Coulomb-Hulthén)-like tensor interaction, is presented in section 2. section 3 is devoted to studying the DDE by applying the usual, well-known Bopp's shift method and the like Greene and Aldrich approximation for the centrifugal term to obtain the effective potentials of the ITSP-ICHLTi model in DDT symmetries. Furthermore, via standard perturbation theory, we find the expectation values of some radial terms to calculate the corrected relativistic energy generated by the effect of the perturbed effective potential of the ITSP-ICHLTi model. We derive the global corrected energy from the ITSP-ICHLTi model. We will also treat the nonrelativistic limit. Section five is devoted to the conclusions.

II. OVERVIEW OF THE EIGENFUNCTIONS AND EIGENVALUES FOE TSP-ICHLTi MODEL IN RQM SYMMETRIES

To construct a physical model describing a physical system that interacted with improved Trigonometric scarf potential within the generalized (Hulthén and Coulomb) like tensor interaction (ITSP-ICHLTi) in deformation Dirac theory, it is useful to recall the eigenvalues and the corresponding eigenfunctions under influence of the corresponding system within the framework of relativistic quantum mechanics known in the literature. In this case, the Trigonometric scarf potential within the generalized (Hulthén and Coulomb) like tensor interaction TSP-ICHLTi is governed by the DE as:

$$\begin{cases} \mathbf{H}_D^{ts}\Psi_{nk}(r,\theta,\varphi)=E_{nk}\Psi_{nk}(r,\theta,\varphi) \\ \mathbf{H}_D^{ts}=\boldsymbol{\alpha}\mathbf{p}+\boldsymbol{\beta}(M+S_{ts}(r))-i\boldsymbol{\beta}\mathbf{r}U(r)+V_{ts}(r) \end{cases} \tag{8}$$

here $\mathbf{H}_D^{ts}$ is the Dirac Hamiltonian operator produced with vector and scalar potentials $(V_{ts}(r),S_{ts}(r))$, M is a reduced rest mass and $\mathbf{p}=-i\hbar\nabla$ it is the momentum operator. The vector potential $V_{ts}(r)$ due to the four-vector linear momentum operator A^{μ} ($V_{ts}(r)$, $\mathbf{A}=0$) and space-time scalar potential $S_{ts}(r)$ due to the mass, E_{nk} is the relativistic eigenvalues, (n,k) representing the principal and spin-orbit coupling terms, respectively. The tensor interaction $U(r)$ equally $\left(-\frac{H_c}{r}-\frac{H_H\exp(-\delta r)}{1-\exp(-\delta r)}\right)$, H_c and H_H are the Coulomb and the Hulthén parameters, $\boldsymbol{\alpha}_i=anti\_diag(\eta_i,\eta_i)$, $\boldsymbol{\beta}=diag(I_{2\times2},-I_{2\times2})$ and η_i are the usual Pauli matrices. Because the Trigonometric scarf potential has spherical symmetry, solutions of the known form

$$\Psi_{nk}(r,\theta,\varphi)=\begin{pmatrix} \frac{F_{nk}(r)}{r}Y_{jm}^{l}(\theta,\varphi) \\ i\,\frac{G_{nk}(r)}{r}Y_{jm^p}^{l^p}(\theta,\varphi) \end{pmatrix},$$

are allowed, here $F_{nk}(r)$ and $G_{nk}(r)$ represent the upper and lower components of the Dirac spinors $\Psi_{nk}(r,\theta,\varphi)$ while $Y_{jm}^{l}(\theta,\varphi)$ and $Y_{jm^p}^{l^p}(\theta,\varphi)$ are the spin and pseudospin spherical harmonics and (m,m^p) are the projections on the z-axis. The upper and lower components, $F_{nk}(r)$ and $G_{nk}(r)$, satisfy the two uncoupled differential equations illustrated below:

$$\begin{aligned}&(\tfrac{d^2}{dr^2}-k(k+1)r^{-2}+U_{eff}^{glt-s}(r)-(M+E_{nk}^{s}-\Delta_{ts}(r))\\ &(M-E_{nk}^{s}+\Sigma_{ts}(r))+\tfrac{\frac{d\Delta_{ts}(r)}{dr}\left(\frac{d}{dr}+\frac{k}{r}-U(r)\right)}{M+E_{nk}^{s}-\Delta_{ts}(r)})F_{nk}(r)=0\end{aligned} \tag{9a}$$

and

$$\begin{aligned}&(\tfrac{d^2}{dr^2}-k(k-1)r^{-2}+U_{eff}^{glt-p}(r)-(M+E_{nk}^{s}-\Delta_{ts}(r))\\ &(M-E_{nk}^{s}+\Sigma_{ts}(r))+\tfrac{\frac{d\Sigma_{ts}(r)}{dr}\left(\frac{d}{dr}-\frac{k}{r}+U(r)\right)}{M+E_{nk}^{s}+\Sigma_{ts}(r)})G_{nk}(r)=0\end{aligned} \tag{9b}$$

Here $U_{eff}^{glt-s}(r)$ is given by:

$$U_{eff}^{glt-s/p}(r)=\begin{cases} \frac{2kU(r)}{r}-\frac{dU(r)}{dr}-U^2(r) \\ \text{for spin sy.} \\ \frac{2kU(r)}{r}+\frac{dU(r)}{dr}-U^2(r) \\ \text{for p/spin sy.} \end{cases} \tag{9c}$$

Putting the tensor interaction $U(r)$ into Eq. (9c) we have:

$$U_{eff}^{glt-s/p}(r)=\begin{cases} \frac{A_1^-}{r^2}-\frac{\delta H_H\exp(-\delta r)}{1-\exp(-\delta r)}+\frac{A_2}{r}\frac{\exp(-\delta r)}{1-\exp(-\delta r)} \\ \quad+\frac{A_3^-\exp(-2\delta r)}{(1-\exp(-\delta r))^2}\text{ for spin sy.} \\ \frac{A_1^+}{r^2}-\frac{\delta H_H\exp(-\delta r)}{1-\exp(-\delta r)}+\frac{A_2}{r}\frac{\exp(-\delta r)}{1-\exp(-\delta r)} \\ \quad+\frac{A_3^+\exp(-2\delta r)}{(1-\exp(-\delta r))^2}\text{ for p/spin sy.} \end{cases} \tag{10}$$

with

$$\begin{cases} A_1^-=-2kH_c-H_c-H_c^2, \\ A_1^+=-2kH_c+H_c-H_c^2, \\ A_2=-2(k+H_c)H_H\ , \\ A_3^-=-(\delta+H_H)H_H, \\ A_3^+=(\delta-H_H)H_H. \end{cases}$$

The negative sign and the positive sign correspond to the effective tensor interaction U_{eff}^{glt-s} and U_{eff}^{glt-p}, respectively, while $\Sigma_{ts}(r)=V_{ts}(r)+S_{ts}(r)$ and $\Delta_{ts}(r)=V_{ts}(r)-S_{ts}(r)$ are determined by:

$$\begin{cases} \Sigma_{ts}(r)=-\frac{V_0}{\sin^2(\alpha r)}\ \text{ and }\ \frac{d\Delta_{ts}(r)}{dr}=0\Rightarrow\Delta_{ts}=C_s \\ \text{for spin sy.} \\ \Delta_{ts}(r)=-\frac{V_0}{\sin^2(\alpha r)}\ \text{ and }\ \frac{d\Sigma_{ts}(r)}{dr}=0\Rightarrow\Sigma_{ts}=C_p \\ \text{for p/spin sy} \end{cases} \tag{11}$$

In RQM symmetries, we derive the following second-order Schrödinger-like equation:

$$\left[\begin{array}{l}\frac{d^2}{dr^2}-k(k+1)r^{-2}+U_{eff}^{glt}(r)\\-\aleph_0\left(M-E_{nk}^{s}+\Sigma_{ts}(r)\right)\end{array}\right]F_{nk}(r)=0\,, \qquad (12)$$

and

$$\left[\begin{array}{l}\frac{d^2}{dr^2}-k(k-1)r^{-2}+U_{eff}^{glt}(r)\\-\left(M+E_{nk}^{p}-\Delta_{ts}(r)\right)\aleph_1\end{array}\right]G_{nk}(r)=0\,, \qquad (13)$$

with $\aleph_0=M+E_{nk}^{s}-C_s$, $\aleph_1=M-E_{nk}^{p}+C_p$ while $k(k-1)$ and $k(k+1)$ are equals $l^p\left(l^p-1\right)$ and $l(l+1)$, respectively. In RQM symmetries, the authors of ref. [7] used the NU method and Greene-Aldrich approximation for the centrifugal term to obtain the expressions of the upper and lower components, $F_{nk}^{s}(s)$ and $G_{nk}^{p}(s)$, as a function of hypergeometric polynomials $P_n^{\left(2\sqrt{\xi_3^s},\,2f_{nk}^{s}\right)}(1-2s)$ and $P_n^{\left(2\sqrt{\lambda_3^p},\,2f_{nk}^{p}\right)}(1-2s)$, respectively,

$$F_{nk}^{s}(s)=N_{nk}^{s}s^{\sqrt{\zeta_3^s}}(1-s)^{\frac{1}{2}+f_{nk}^{s}}P_n^{\left(2\sqrt{\xi_3^s},\,2f_{nk}^{s}\right)}(1-2s)\,, \qquad (14)$$

and

$$G_{nk}^{p}(s)=N_{nk}^{ps}s^{\sqrt{\lambda_3^p}}(1-s)^{\frac{1}{2}+f_{nk}^{p}}P_n^{\left(2\sqrt{\lambda_3^p},\,2f_{nk}^{p}\right)}(1-2s)\,, \qquad (15)$$

here $s=\exp(-2\alpha r)$ while f_{nk}^{s} and f_{nk}^{p} are given by:

$$\left.\begin{array}{l}f_{nk}^{s}=\sqrt{\frac{1}{4}+\zeta_1^s-\zeta_3^s+\frac{H_H\delta^{-1}-4\aleph_0V_0}{\delta^2}},\\ f_{nk}^{p}=\sqrt{\frac{1}{4}+\lambda_1^p-\lambda_3^p-\frac{H_H\delta^{-1}-4\aleph_1V_0}{\delta^2}}.\end{array}\right\}$$

while ζ_1^s, ζ_3^s, λ_1^p and λ_3^p are given by:

$$\left.\begin{array}{l}\zeta_1^s=(k+H_c)(k+H_c+1)+\frac{H_H}{\delta}\left(2k+2H_H+\frac{H_H}{\delta}\right)+\zeta_3^s\\ \zeta_3^s=\frac{\aleph_0\left(M-E_{nk}^s\right)}{\delta^2},\ \lambda_3^p=\frac{\aleph_1\left(M+E_{nk}^s\right)}{\delta^2}\\ \lambda_1^p=(k+H_c)(k+H_c-1)+\frac{H_H}{\delta}\left(2k+2H_H-\frac{H_H}{\delta}\right)+\lambda_3^p\end{array}\right\} \qquad (16)$$

while N_{nk}^{s} and N_{nk}^{ps} are the normalization constants. The equations of energy for the spin symmetry and the p-spin symmetry, are given by [7]:

$$\frac{\aleph_0\left(M-E_{nk}^{s}\right)}{\delta^2}=\frac{n(n+1)+\frac{\delta^2+2H_H\delta^{-1}-8\aleph_0V_0}{2\delta^2}+\left(n+\frac{1}{2}\right)\sqrt{1+4\left(\zeta_1^s+H_H\delta^{-1}-4\aleph_0V_0\delta^{-2}\right)}}{1+2n+\sqrt{1+4\left(\zeta_1^s+H_H\delta^{-1}-4\aleph_0V_0\delta^{-2}\right)}} \qquad (17)$$

and

$$\frac{\aleph_1\left(M+E_{nk}^{p}\right)}{\delta^2}=\frac{n(n+1)+\frac{\delta^2+2H_H\delta^{-1}+8\aleph_1V_0}{2\delta^2}+\left(n+\frac{1}{2}\right)X(k,\delta,V_0,H_c,H_H)}{1+2n+X(k,\delta,V_0,H_c,H_H)}\,, \qquad (18)$$

where,

$$X(k,\delta,V_0,H_c,H_H)=\sqrt{1+4\left(\lambda_1^p-2(k+H_c)-H_H\delta^{-1}+4\aleph_1V_0\delta^{-2}\right)}.$$

In the following sections, we will need another formula later for the upper and lower components $F_{nk}^{s}(s)$ and $G_{nk}^{p}(s)$. We will use the standard transform expression $P_n^{(a_n,b_n)}(1-2s)$ known in the literature, in the following form:

$$P_n^{(a_n,b_n)}(1-2s)=\frac{\Gamma(n+a_n+1)}{n!\Gamma(a_n+1)}\,{}_2F_1(-n,n+a_n+b_n+1;1+a_n,s) \qquad (19)$$

Thus, he upper $F_{nk}^{s}(s)$ and lower $G_{nk}^{p}(s)$ components for ITSP-ICHLTi become,

$$F_{nk}^{s}(s)=N_{nk}^{ns}s^{\sqrt{\zeta_3^s}}(1-s)^{\frac{1}{2}+f_{nk}^{s}}{}_2F_1\left(-n,n+2\sqrt{\xi_3^s}+2f_{nk}^{s}+1;1+2\sqrt{\xi_3^s},s\right) \qquad (20a)$$

and

$$G_{nk}^{p}(s)=N_{nk}^{np}s^{\sqrt{\lambda_3^p}}(1-s)^{\frac{1}{2}+f_{nk}^{p}}{}_2F_1\left(-n,n+2\sqrt{\lambda_3^p}+2f_{nk}^{p}+1;1+2\sqrt{\lambda_3^p},s\right) \qquad (20b)$$

here N_{nk}^{ns} and N_{nk}^{np} are equal $N_{nk}^{s}\frac{\Gamma\left(n+2\sqrt{\xi_3^s}+1\right)}{n!\Gamma\left(2\sqrt{\xi_3^s}+1\right)}$ and $N_{nk}^{ps}\frac{\Gamma\left(n+a_n2\sqrt{\lambda_3^p}+1\right)}{n!\Gamma\left(2\sqrt{\lambda_3^p}+1\right)}$, respectively. The lower component $G_{nk}^{s}(s)$ of spin symmetry and the upper component $F_{nk}^{p}(s)$ of p-spin symmetry are obtained as follows:

$$G_{nk}^{s}(s)=\frac{1}{M+E_{nk}^{s}-C_s}\left(\frac{d}{dr}+\frac{k}{r}-U^{glt}(r)\right)F_{nk}^{s}(s)\,, \qquad (21)$$

and

$$F_{nk}^{p}(s)=\frac{1}{M-E_{nk}^{p}+C_p}\left(\frac{d}{dr}+\frac{k}{r}-U^{glt}(r)\right)G_{nk}^{p}(s)\,. \qquad (22)$$

III. The NEW SOLUTIONS OF DDE UNDER THE ITSP-ICHLTi IN THE DDT SYMMETRIES:

A Review of Bopp's shift method

In this subsection, let us begin by finding the DDE in the symmetries of the deformation Dirac theory under the ITSP-ICHLTi model. Our objective is achieved by applying the new principles which we have seen in the introduction, Eqs. (4) and (7), summarized in the new relationships MASCCCRs and the notion of the Weyl-Moyal star product. In the DDT symmetry, these data will allow us to rewrite the upper and lower components, $F_{nk}(r)$ and $G_{nk}(r)$ in Eqs. (12) and (13) as follows:

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k+1)r^{-2} + U_{eff}^{glt-s}(r) \\ -\aleph_0\left(M - E_{nk}^{s} + \Sigma_{ts}(r)\right) \end{bmatrix} * F_{nk}^{s}(r) = 0\,, \tag{23}$$

and

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k-1)r^{-2} + U_{eff}^{glt-p}(r) \\ -\left(M + E_{nk}^{p} - \Delta_{ts}(r)\right)\aleph_1 \end{bmatrix} * G_{nk}^{p}(r) = 0\,. \tag{24}$$

The application of the Connes technique [24, 25], or the Seiberg and Witten [26] are two alternative ways to obtain answers to Eqs. (23) and (24). Below, we will introduce the simple but powerful Bopp's shift method which has solved many vital physical problems in deformation quantum mechanics; the details can be found in [8, 9, 10, 11]. It is known to specialists that the star product can be translated into the ordinary product known in the literature using what is called Bopp's shift method. F. Bopp was the first to consider pseudo-differential operators derived from a symbol using the quantization rules (x and p) $\rightarrow$ ($x_{nc} = x$ and $p_{nc} = p + \frac{i}{2}\partial_x$) rather than the ordinary correspondence: (x and p) $\rightarrow$ ($x_{nc} = x$ and $p_{nc} = p + \frac{i}{2}\partial_x$), respectively. For the researchers, this procedure is known as Bopp's shifts, and this quantization procedure is known as Bopp quantization. In recent years, this method has had a great deal of success. Under the influence of a variety of potentials, researchers were looking for a solution to the deformed nonrelativistic Schrödinger equation (DNRSE), this is through their successful application of Bopp's shift method (see for example some typical references [58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69]). On the other hand, this method has achieved other successes on a relativistic level, for example, we find some typical applications of this method in the framework of the deformed relativistic Klein-Gordon equation (DRKGE) (see the refs. [28, 29, 30, 71, 72, 73, 74]), for the DDE (see for example the Refs. [34, 75, 76, 77, 78, 79, 80]) and for the deformed relativistic Duffin-Kemmer-Petiau equation (DRDKPE) [81, 82]. Thus, Bopp's shift method is based on reducing second-order linear differential equations of the DNRSE, DRKGE, DDE, and DRDKPE with the Weyl-Moyal star product to second-order linear differential equations of NRSE, RKGE, DE, and RDKPE without the Weyl-Moyal star product with simultaneous translation in the space-space. It is worth noting that Bopp's shift method allows us to reduce Eqs. (23) and (24) to their simplest form with the ordinary product known in the litterateur:

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k+1)r_{nc}^{-2} + U_{eff}^{glt-s}(r_{nc}) \\ -\aleph_0\left(M - E_{nk}^{s} + \Sigma_{ts}(r_{nc})\right) \end{bmatrix} F_{nk}^{s}(r) = 0\,, \tag{25}$$

and

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k-1)r_{nc}^{-2} + U_{eff}^{glt-p}(r_{nc}) \\ -\left(M + E_{nk}^{p} - \Delta_{ts}(r_{nc})\right)\aleph_1 \end{bmatrix} G_{nk}^{p}(r) = 0\,. \tag{26}$$

The modified algebraic structure of covariant canonical commutation relations with the notion of the Weyl-Moyal star product in Eqs. (4) become new METNCCCRs with ordinary known products in literature is as follows (see, e.g., [8, 9, 10, 11]):

$$\begin{cases} \left[x_{nc\mu}^{(s,h,i)}, p_{nc\nu}^{(s,h,i)}\right] = i\hbar_{eff}\delta_{\mu\nu} \\ \left[x_{nc\mu}^{(s,h,i)}, x_{nc\nu}^{(s,h,i)}\right] = i\theta_{\mu\nu} \end{cases} \tag{27}$$

In the symmetries of DDT, the generalized positions and momentum coordinates ($x_{nc\mu}^{(s,h,i)}$ and $p_{nc\mu}^{(s,h,i)}$) are defined as [8-11]:

$$\begin{cases} x_{nc\mu}^{(s,h,i)} = x_{\mu}^{(s,h,i)} - \sum_{\nu=1}^{3} \frac{i\theta_{\mu\nu}}{2} p_{\nu}^{(s,h,i)} \\ p_{nc\mu}^{(s,h,i)} = p_{\mu}^{(s,h,i)} \end{cases} \tag{28}$$

For spin symmetry and p-spin symmetry, respectively, the operator r_{nc}^2 equal ($r^2 - \mathbf{L\Theta}$ and $r^2 - \mathbf{L^p\Theta}$) [34, 75, 76, 77, 78, 79, 80], while the new operators $V_{ts}(r_{nc})$, $U_{eff}^{glt}(r_{nc})$, $k(k+1)r_{nc}^{-2}$ and $k(k-1)r_{nc}^{-2}$ in the DDT symmetries, are expressed as:

$$\begin{cases} V_{ts}^{s}(r_{nc}) = -\frac{V_0}{\sin^2(\alpha r)} - \frac{\partial V_{ts}(r)}{\partial r}\frac{\mathbf{L\Theta}}{2r} + O\left(\Theta^2\right) \\ V_{ts}^{p}(r_{nc}) = -\frac{V_0}{\sin^2(\alpha r)} - \frac{\partial V_{ts}(r)}{\partial r}\frac{\mathbf{L}^p\mathbf{\Theta}}{2r} + O\left(\Theta^2\right) \\ U_{eff}^{glt^{-s}}(r_{nc}) = \frac{2kU(r)}{r} - \frac{dU(r)}{dr} - U^2(r) \\ -\frac{\partial U_{eff}^{glt-s}(r)}{\partial r}\frac{\mathbf{L\Theta}}{2r} + O\left(\Theta^2\right) \end{cases} \tag{29a}$$

and

$$\begin{cases} U_{eff}^{glt-p}(r_{nc}) = \frac{2kU(r)}{r} + \frac{dU(r)}{dr} - U^2(r) \\ -\frac{\partial U_{eff}^{glt-p}(r)}{\partial r}\frac{\mathbf{L}^p\Theta}{2r} + O(\Theta^2) \\ k(k+1)r_{nc}^{-2} = k(k+1)r^{-2} + k(k+1)r^{-4}\mathbf{L}\Theta + O(\Theta^2) \\ k(k-1)r_{nc}^{-2} = k(k-1)r^{-2} + k(k-1)r^{-4}\mathbf{L}^p\Theta + O(\Theta^2) \end{cases} \tag{29b}$$

Substituting Eqs. (29a) and (29b) into Eqs. (25) and (26), we obtain the following two similar Schrödinger equations:

$$\left(\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} + U_{eff}^{glt-s}(r) - \aleph_0\left(M - E_{nk}^s + \Sigma_{ts}(r)\right) - \Xi_{ts}^{pert}(r)\right)F_{nk}^s(r) = 0, \tag{30}$$

and

$$\left(\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} + U_{eff}^{glt-p}(r) - \left(M + E_{nk}^p - \Delta_{ts}(r)\right)\aleph_1 - \Upsilon_{ts}^{pert}(r)\right)G_{nk}^p(r) = 0, \tag{31}$$

with

$$\Xi_{ts}^{pert}(r) = \left(-\frac{1}{2r}\frac{\partial\left(U_{eff}^{glt-s}(r) + \aleph_0 V_{ts}(r)\right)}{\partial r} + \frac{k(k+1)}{r^4}\right)\mathbf{L}\Theta + O(\Theta^2), \tag{32}$$

and

$$\Upsilon_{ts}^{pert}(r) = \left(-\frac{1}{2r}\frac{\partial\left(U_{eff}^{glt-p}(r) + \aleph_1 V_{ts}(r)\right)}{\partial r} + \frac{k(k-1)}{r^4}\right)\mathbf{L}^p\Theta + O(\Theta^2). \tag{33}$$

By comparing (Eqs. (12) and (13)) and (Eqs. (30) and (31)), we observe two additive potentials ($\Xi_{ts}^{pert}(r)$ and $\Upsilon_{ts}^{pert}(r)$). Moreover, these terms are proportional to the infinitesimal noncommutativity parameter Θ. From a physical point of view, this means that these two spontaneously generated terms ($\Xi_{ts}^{pert}(r)$ and $\Upsilon_{ts}^{pert}(r)$) as a result of the topological properties of the deformation space-space can be considered very small compared to the fundamental terms ($\Sigma_{ts}(r)$ and $\Delta_{ts}(r)$), respectively. A direct calculation gives ($\frac{\partial V_{ts}(r)}{\partial r}$ and $\frac{\partial U_{eff}^{glt-s/p}(r)}{\partial r}$) as follows:

$$\frac{\partial V_{ts}(r)}{\partial r} = \frac{2V_0}{\alpha^2}\frac{1}{r^3}, \tag{34a}$$

and

$$\begin{aligned} \frac{\partial U_{eff}^{glt-s/p}(r)}{\partial r} &= -\frac{2A_1^{\mp}}{r^3} + \frac{\delta^2 H_H \exp(-\delta r)}{1-\exp(-\delta r)} \\ &+ \frac{\left(\delta^2 H_H - 2\delta A_3^{\mp}\right)\exp(-2\delta r)}{(1-\exp(-\delta r))^2} - \frac{A_2}{r^2}\frac{\exp(-\delta r)}{1-\exp(-\delta r)} \\ &- \frac{\delta A_2}{r}\frac{\exp(-\delta r)}{1-\exp(-\delta r)} - \frac{\delta A_2}{r}\frac{\exp(-2\delta r)}{(1-\exp(-\delta r))^2} \\ &- \frac{2\delta A_3^{\mp}\exp(-3\delta r)}{(1-\exp(-\delta r))^3} \end{aligned} \tag{34b}$$

Substituting Eqs. (34a) and (34b) into Eqs. (32) and (33), we obtain spontaneously generated terms ($\Xi_{ts}^{pert}(r)$ and $\Upsilon_{ts}^{pert}(r)$) as follows:

$$\begin{aligned} \Xi_{ts}^{pert}(r) = \\ \Big(\frac{k(k+1) + A_1^+ - \frac{V_0\aleph_0}{\alpha^2}}{r^4} &+ \frac{1}{2r}\frac{\delta^2 H_H \exp(-\delta r)}{1-\exp(-\delta r)} \\ -\frac{1}{2r}\frac{\left(\delta^2 H_H - 2\delta A_3\right)\exp(-2\delta r)}{(1-\exp(-\delta r))^2} &+ \frac{A_2}{2r^3}\frac{\exp(-\delta r)}{1-\exp(-\delta r)} \\ +\frac{\delta A_2}{2r^2}\frac{\exp(-\delta r)}{1-\exp(-\delta r)} &+ \frac{\delta A_2}{2r^2}\frac{\exp(-2\delta r)}{(1-\exp(-\delta r))^2} \\ +\frac{1}{2r}\frac{2\delta A_3^+\exp(-3\delta r)}{(1-\exp(-\delta r))^3}\Big)\mathbf{L}\Theta + O(\Theta^2) \end{aligned} \tag{35}$$

and

$$\begin{aligned} \Upsilon_{ts}^{pert}(r) = \Xi_{ts}^{pert}(r)(\aleph_0 \to \aleph_1, k(k+1) \to k(k-1), \\ \left(A_1^+, A_3^+\right) \to \left(A_1^-, A_3^-\right) \text{ and } \mathbf{L}\Theta \to \mathbf{L}^p\Theta) \end{aligned} \tag{36}$$

Furthermore, using the unit step function (also known as the Heaviside step function $\theta(y)$ or simply the theta function) we can rewrite the global induced two potentials ($\Sigma_{t\_ts}^{pert}(r)$ and $\Delta_{t-ts}^{pert}(r)$) for a spin and p-spin symmetries corresponding upper and lower components ($F_{nk}^s(s)$ and $G_{nk}^s(s)$) and ($F_{nk}^p(s)$ and $G_{nk}^p(s)$), respectively as:

$$\begin{aligned} \Sigma_{t\_ts}^{pert}(r) &= \Xi_{ts}^{pert}(r)\theta\left(E_{nc}^{ts-s}\right) - \Xi_{ts}^{pert}(r)\theta\left(-E_{nc}^{ts-s}\right) \\ &= \begin{cases} \Xi_{ts}^{pert}(r) \text{ for UC of spin sy.} \\ -\Xi_{ts}^{pert}(r) \text{ for LC of spin sy.} \end{cases} \end{aligned} \tag{37a}$$

and

$$\begin{aligned} \Delta_{t\_ts}^{pert}(r) &= \Upsilon_{ts}^{pert}(r)\theta\left(E_{nc}^{ts-p}\right) - \Upsilon_{ts}^{pert}(r)\theta\left(-E_{nc}^{ts-p}\right) \\ &= \begin{cases} \Upsilon_{ts}^{pert}(r) \text{ for UC of p-spin sy.} \\ -\Upsilon_{ts}^{pert}(r) \text{ for LC of p-spin sy.} \end{cases} \end{aligned} \tag{37b}$$

Here UC and LC are the upper components and lower components. The step function $\theta(y)$ is given by:

$$\theta(x) = \begin{cases} 1 \text{ for } y \geq 0 \\ 0 \text{ for } y \langle 0 \end{cases} \tag{37c}$$

For spin symmetry, we first consider Eq. (30), which contains the improved Trigonometric scarf potential within the generalized (Hulthén and Coulomb)-like tensor interaction in the deformation of Dirac theory symmetries. It can be solved exactly only for $k=0$ and $k=-1$ in the absence of like tensor interaction ($H_c = 0$ and $H_H = 0$) since the two centrifugal terms (proportional to $k(k+1)r^{-2}$ and $k(k+1)r^{-4}$) vanish. In the case of arbitrary k, an appropriate approximation should be employed on the centrifugal terms. We apply the following improved approximation which was applied by Greene and Aldrich [83]:

$$\frac{1}{r^2}\approx\frac{\delta^2}{\left(1-e^{-\delta r}\right)^2}=\frac{\delta^2}{(1-s)^2}\Leftrightarrow\frac{1}{r}\approx\frac{\delta}{1-e^{-\delta r}}=\frac{\delta}{1-s}, \quad (38)$$

with $\delta=2\alpha$. It should be noted that many researchers have used this approximation in relativistic and non-relativistic cases [84, 85, 86]. For p-spin symmetry, we now consider Eq. (31) and will follow similar steps as in the spin symmetry case in the deformation of Dirac theory symmetries. Same as before, Eq. (31) cannot be solved exactly for $k=0$ and $k=1$ without tensor interaction, since the two centrifugal terms (proportional to $k(k-1)r^{-2}$ and $k(k-1)r^{-4}$). Applying the approximations Eq. (37) to the centrifugal terms of Eqs. (35) and (36), the general form of the additive potentials $\Xi_{ts}^{pert}(s)$ and $\Upsilon_{ts}^{pert}(s)$ will be as follows:

$$\begin{aligned}\Xi_{ts}^{pert}(r)=(&\tfrac{T_{nk}^{1s}}{(1-s)^4}+\tfrac{T_{nk}^{2s}s}{(1-s)^2}+\tfrac{T_{nk}^{3s}s^2}{(1-s)^3}+\tfrac{T_{nk}^{4s}s}{(1-s)^4}\\&+\tfrac{T_{nk}^{5s}s}{(1-s)^3}+\tfrac{T_{nk}^{6s}s^2}{(1-s)^4}+\tfrac{T_{nk}^{7s}s^3}{(1-s)^4})\mathbf{L}\Theta+O\left(\Theta^2\right)\end{aligned} \quad (39a)$$

and

$$\begin{aligned}\Upsilon_{ts}^{pert}(s)=(&\tfrac{T_{nk}^{1p}}{(1-s)^4}+\tfrac{T_{nk}^{2p}s}{(1-s)^2}+\tfrac{T_{nk}^{3p}s^2}{(1-s)^3}+\tfrac{T_{nk}^{4p}s}{(1-s)^4}\\&+\tfrac{T_{nk}^{5p}s}{(1-s)^3}+\tfrac{T_{nk}^{6p}s^2}{(1-s)^4}+\tfrac{T_{nk}^{7p}s^3}{(1-s)^4}\mathbf{L}^p\Theta+O\left(\Theta^2\right)\end{aligned} \quad (39b)$$

with

$$\begin{cases}T_{nk}^{1s}=\left(k(k+1)+A_1^+-\frac{V_0\aleph_0}{\alpha^2}\right)\delta^4,\\ T_{nk}^{1p}=\left(k(k-1)+A_1^--\frac{V_0\aleph_1}{\alpha^2}\right)\delta^4,\\ T_{nk}^{2s}=T_{nk}^{2p}=\frac{\delta^3H_H}{2},T_{nk}^{3s}=-\frac{\left(\delta H_H-2A_3^-\right)\delta^2}{2},\\ T_{nk}^{3p}=-\frac{\left(\delta H_H-2A_3^+\right)\delta^2}{2},T_{nk}^{4s}=T_{nk}^{4p}=\frac{A_2\delta^3}{2},\\ T_{nk}^{5s}=T_{nk}^{6s}=T_{nk}^{5p}=T_{nk}^{6p}=\frac{\delta^3A_2}{2},\\ T_{nk}^{7s}=\delta^2A_3^-\text{ and }T_{nk}^{7p}=\delta^2A_3^+.\end{cases} \quad (40)$$

Notably, the results yielded by the Greene and Aldrich approximation, for small values $\delta r<<1$, are in good agreement with those obtained using other methods. We have replaced the terms (($k(k+1)r^{-4}$ and $k(k-1)r^{-4}$) with the approximation in Eq. (38). The trigonometric scarf potential including a generalized (Coulomb-Hulthén)-like tensor interaction is extended by including new additive potentials $\Xi_{ts}^{pert}(r)$ and $\Upsilon_{ts}^{pert}(r)$ expressed to the radial terms:

$$\left\{\begin{matrix}\frac{1}{(1-s)^4},\frac{s}{(1-s)^2},\frac{s^2}{(1-s)^3},\frac{s}{(1-s)^4},\\ \frac{s}{(1-s)^3},\frac{s^2}{(1-s)^4}\text{and}\frac{s^3}{(1-s)^4}.\end{matrix}\right\} \quad (41)$$

to become the improved Trigonometric scarf potential including an improved generalized (Coulomb-Hulthén)-like tensor interaction in DDT symmetries. The newly generated two effective potentials ($\Xi_{ts}^{pert}(r)$ and $\Upsilon_{ts}^{pert}(r)$) are also proportional to the infinitesimal vector Θ. This allows us to consider the new additive parts of the effective potential ($\Xi_{ts}^{pert}(r)$ and $\Upsilon_{ts}^{pert}(r)$) as perturbation potentials compared to the main potentials $\Sigma_{ts}(r)$ and $\Delta_{ts}(r)$ which are also known with the parent potential operator in the symmetries of DDT, that is, the two inequalities ($\Xi_{ts}^{pert}(r)<<\Sigma_{ts}(r)$ and $\Upsilon_{ts}^{pert}(r)<<\Delta_{ts}(r)$) have become achieved. That is all physical justifications for applying the time-independent perturbation theory become satisfied to calculate the expectation values of previous radial terms. This allows us to give a complete prescription for determining the energy level of the generalized $\left(n,l,m,l^p,m^p,s,s^p\right)^{th}$ excited states.

B The expectation values under the ITSP-ICHLTi in the DDT for spin symmetry

In this subsection, we want to apply the perturbative theory, in the case of deformation Dirac theory symmetries, we find the expectation values $M_{1(nlms)}^{s-ts}, M_{2(nlms)}^{s-ts}, M_{3(nlms)}^{s-ts}, M_{4(nlms)}^{s-ts}$, $M_{5(nlms)}^{s-ts}, M_{6(nlms)}^{s-ts}$ and $M_{7(nlms)}^{s-ts}$ which are equals $\left\langle\frac{1}{(1-s)^4}\right\rangle_{(nlms)}^{sp-ts}$, $\left\langle\frac{s}{(1-s)^2}\right\rangle_{(nlms)}^{sp-ts}$, $\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(nlms)}^{sp-ts}$, $\left\langle\frac{s}{(1-s)^4}\right\rangle_{(nlms)}^{s-ts}$, $\left\langle\frac{s}{(1-s)^3}\right\rangle_{(nlms)}^{s-ts}$, $\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(nlms)}^{s-ts}$ and $\left\langle\frac{s^3}{(1-s)^4}\right\rangle_{(nlms)}^{s-ts}$ respectively, for the spin symmetry taking into account the wave function which we have seen previously in Eq. (14a). Thus, after straightforward calculations, we obtain the following results:

$$M_{1(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+\infty}s^{2\sqrt{\zeta_3^s}}(1-s)^{2f_{nk}^s-3}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2dr \quad (42a)$$

$$M_{2(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+\infty}s^{2\sqrt{\zeta_3^s}+1}(1-s)^{2f_{nk}^s-1}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2dr \quad (42b)$$

$$M_{3(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+\infty}s^{2\sqrt{\zeta_3^s}+2}(1-s)^{2f_{nk}^s-2}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2dr \quad (42c)$$

$$M_{4(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+\infty}s^{2\sqrt{\zeta_3^s}+1}(1-s)^{2f_{nk}^s-3}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2dr \quad (42d)$$

$$M_{5(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+\infty}s^{2\sqrt{\zeta_3^s}+1}(1-s)^{2f_{nk}^s-2}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2dr \tag{42e}$$

$$M_{6(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+\infty}s^{2\sqrt{\zeta_3^s}+2}(1-s)^{2f_{nk}^s-3}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2dr \tag{42f}$$

$$M_{7(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+\infty}s^{2\sqrt{\zeta_3^s}+3}(1-s)^{2f_{nk}^s-3}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2dr \tag{42j}$$

the values Λ_{new}^s equal $n+2\sqrt{\xi_3^s}+2f_{nk}^s+1$. We have used useful abbreviations $\langle M\rangle_{(nlms)}^{s-ts}=\langle n,l,m,s|M|n,l,m,s\rangle$ to avoid the extra burden of writing equations. Furthermore, we have applied the property of spherical harmonics, which has the form

$$\int Y_l^m(\theta',\varphi')Y_{l'}^{m'}(\theta,\varphi)\sin(\theta)d\theta d\varphi=\delta_{ll'}\delta_{mm'}$$

Introducing the change of variable $s=\exp(-\delta r)$. This maps the region $0\le r\prec\infty$ to $0\le s\le 1$ and allows us to obtain $dr=-\frac{ds}{s}$, and transform Eqs. (42a, 42b, 42c, 42d, 42e, 42f and 42j) in the following form:

$$M_{1(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+1}s^{2\sqrt{\zeta_3^s}-1}(1-s)^{2f_{nk}^s-2-1}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2ds \tag{43a}$$

$$M_{2(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+1}s^{2\sqrt{\zeta_3^s}+1-1}(1-s)^{2f_{nk}^s-1}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2ds \tag{43b}$$

$$M_{3(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+1}s^{2\sqrt{\zeta_3^s}+2-1}(1-s)^{2f_{nk}^s-1-1}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2ds \tag{43c}$$

$$M_{4(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+1}s^{2\sqrt{\zeta_3^s}+1-1}(1-s)^{2f_{nk}^s-2-1}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2ds \tag{43d}$$

$$M_{5(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+1}s^{2\sqrt{\zeta_3^s}+1-1}(1-s)^{2f_{nk}^s-1-1}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2ds \tag{43e}$$

$$M_{6(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+1}s^{2\sqrt{\zeta_3^s}+2-1}(1-s)^{2f_{nk}^s-2-1}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2ds \tag{43f}$$

$$M_{7(nlms)}^{s-ts}=N_{nk}^{ns2}\int_0^{+1}s^{2\sqrt{\zeta_3^s}+3-1}(1-s)^{2f_{nk}^s-2-1}\left[{}_2F_1\left(-n,\Lambda_{new}^s;1+2\sqrt{\xi_3^s},s\right)\right]^2ds \tag{43j}$$

We can evaluate the above integrals either in a recurrence way through the physical values of the principal quantum number ($n=0,1,...$) and then generalize the result to the general $(n,l,m,s)^{th}$ excited state or we use the method proposed by Dong *et al.* [87] and applied by Zhang [88], to obtain the general excited state directly. We calculate the integrals in Eqs. (43a, 43b, 43c, 43d, 43e, 43f and 43j) with the help of the special integral formula:

$$\int_0^{+1}s^{\xi-1}(1-s)^{\eta-1}\left[{}_2F_1(c_1,c_2;c_3;s)\right]^2ds=\frac{\Gamma(\xi)\Gamma(\eta)}{\Gamma(\xi+\eta)}\,{}_3F_2(c_1,c_2,\eta;c_3,\eta+\xi;1) \tag{44}$$

here ${}_2F_1(c_1,c_2;c_3;z)$ is the generalized hypergeometric function and ${}_3F_2(c_1,c_2,\eta;c_3,\eta+\xi;1)$ equal $\sum_{n=0}^{+\infty}\frac{(c_1)_n(c_2)_n(\eta)_n}{(c_3)_n(\eta+\xi)n!}$, here $(c_1)_n$ denote to the rising factorial or Pochhammer symbol while $\Gamma(\xi)$ denoting the usual Gamma function. By identifying Eq. (44) with the integrals, we obtain the following results:

$$M_{1(nlms)}^{s-ts}=N_{nk}^{ns2}\frac{\Gamma\left(2\sqrt{\zeta_3^s}\right)\Gamma\left(2f_{nk}^s-2\right)}{\Gamma\left(T_{nk}^s-2\right)} \tag{45a}$$
$${}_3F_2\left(-n,X_{nk}^s,2f_{nk}^s-2;1+2\sqrt{\xi_3^s},T_{nk}^s-2;1\right)$$

$$M_{2(nlms)}^{s-ts}=N_{nk}^{ns2}\frac{\Gamma\left(2\sqrt{\zeta_3^s}+1\right)\Gamma\left(2f_{nk}^s\right)}{\Gamma\left(T_{nk}^s+1\right)} \tag{45b}$$
$${}_3F_2\left(-n,X_{nk}^s,2f_{nk}^s;1+2\sqrt{\xi_3^s},T_{nk}^s+1;1\right)$$

$$M_{3(nlms)}^{s-ts}=N_{nk}^{ns2}\frac{\Gamma\left(2\sqrt{\zeta_3^s}+2\right)\Gamma\left(2f_{nk}^s-1\right)}{\Gamma\left(T_{nk}^s+1\right)} \tag{45c}$$
$${}_3F_2\left(-n,X_{nk}^s,2f_{nk}^s-1;1+2\sqrt{\xi_3^s},T_{nk}^s+1;1\right)$$

$$M_{4(nlms)}^{s-ts}=N_{nk}^{ns2}\frac{\Gamma\left(2\sqrt{\zeta_3^s}+1\right)\Gamma\left(2f_{nk}^s-2\right)}{\Gamma\left(T_{nk}^s-1\right)} \tag{45d}$$
$${}_3F_2\left(-n,X_{nk}^s,2f_{nk}^s-2;1+2\sqrt{\xi_3^s},T_{nk}^s-1;1\right)$$

$$M_{5(nlms)}^{s-ts} = N_{nk}^{ns2} \frac{\Gamma\left(2\sqrt{\zeta_3^s}+1\right)\Gamma\left(2f_{nk}^s-1\right)}{\Gamma\left(T_{nk}^s\right)} \tag{45e}$$

$${}_3F_2\left(-n, X_{nk}^s, 2f_{nk}^s-1; 1+2\sqrt{\xi_3^s}, T_{nk}^s; 1\right),$$

$$M_{6(nlms)}^{s-ts} = N_{nk}^{ns2} \frac{\Gamma\left(2\sqrt{\zeta_3^s}+2\right)\Gamma\left(2f_{nk}^s-2\right)}{\Gamma\left(T_{nk}^s\right)} \tag{45f}$$

$${}_3F_2\left(-n, X_{nk}^s, 2f_{nk}^s-2; 1+2\sqrt{\xi_3^s}, T_{nk}^s; 1\right),$$

$$M_{7(nlms)}^{s-ts} = N_{nk}^{ns2} \frac{\Gamma\left(2\sqrt{\zeta_3^s}+3\right)\Gamma\left(2f_{nk}^s-2\right)}{\Gamma\left(T_{nk}^s+1\right)} \tag{45j}$$

$${}_3F_2\left(-n, X_{nk}^s, 2f_{nk}^s-2; 1+2\sqrt{\xi_3^s}, T_{nk}^s+1; 1\right),$$

with X_{nk}^s and T_{nk}^s are equal $2\sqrt{\xi_3^s}+2f_{nk}^s+n+1$ and $2\sqrt{\xi_3^s}+2f_{nk}^s$ respectively.

C The expectation values under the ITSP-ICHLTi in the DDT for p-spin symmetry

In this subsection, we want to apply the perturbative theory, in the case of deformation Dirac theory symmetries, we find the expectation values: $M_{1(nl^p m^p s^p)}^{p-ts}$, $M_{2(nl^p m^p s^p)}^{p-ts}$, $M_{3(nl^p m^p s^p)}^{p-ts}$, $M_{4(nl^p m^p s^p)}^{p-ts}$, $M_{5(nl^p m^p s^p)}^{p-ts}$, $M_{6(nl^p m^p s^p)}^{p-ts}$ and $M_{7(nl^p m^p s^p)}^{p-ts}$ which are equal $\left\langle \frac{1}{(1-s)^4} \right\rangle_{(nl^p m^p s^p)}^{p-ts}$, $\left\langle \frac{s}{(1-s)^2} \right\rangle_{(nl^p m^p s^p)}^{p-ts}$, $\left\langle \frac{s^2}{(1-s)^3} \right\rangle_{(nl^p m^p s^p)}^{p-ts}$, $\left\langle \frac{s}{(1-s)^4} \right\rangle_{(nl^p m^p s^p)}^{p-ts}$, $\left\langle \frac{s}{(1-s)^3} \right\rangle_{(nl^p m^p s^p)}^{p-ts}$, $\left\langle \frac{s^2}{(1-s)^4} \right\rangle_{(nl^p m^p s^p)}^{p-ts}$ and $\left\langle \frac{s^3}{(1-s)^4} \right\rangle_{(nl^p m^p s^p)}^{p-ts}$, respectively for p-spin symmetry with tensor interaction taking into account the wave function which we have seen previously in Eq. (14a). On a careful inspection of the upper wave function $F_{nk}(r)$ and the lower wave function $G_{nk}(r)$ in Sec. 3, we discovered that the upper component $F_{nk}(r)$ can be transformed into the lower component $G_{nk}(r)$ and vice versa. This can be achieved by using the following transformations in the $\left(N_{nk}^p, \lambda_3^p, f_{nk}^p\right)$ and $\left(N_{nk}^s, \zeta_3^s, f_{nk}^s\right)$ (in Eqs. (15) and (16)):

$$\begin{cases} F_{nk}(r) \Leftrightarrow G_{nk}(r) \Rightarrow N_{nk}^p \Leftrightarrow N_{nk}^s, \\ \sqrt{\lambda_3^p} \Leftrightarrow \sqrt{\zeta_3^s} \text{ and } f_{nk}^p \Leftrightarrow f_{nk}^s. \end{cases} \tag{46}$$

This allows us to obtain the expectation values for p-spin symmetry from Eqs. (45a, 45b, 45c, 45d, 45e, 45f and 45j) without re-calculation, as follows:

$$M_{1(nl^p m^p s^p)}^{p-ts} = N_{nk}^{np2} \frac{\Gamma\left(2\sqrt{\lambda_3^p}\right)\Gamma\left(2f_{nk}^s-2\right)}{\Gamma\left(T_{nk}^s-2\right)} \tag{47a}$$

$${}_3F_2\left(-n, X_{nk}^s, 2f_{nk}^s-2; 1+2\sqrt{\lambda_3^p}, T_{nk}^s-2; 1\right)$$

$$M_{2(nl^p m^p s^p)}^{p-ts} = N_{nk}^{np2} \frac{\Gamma\left(2\sqrt{\lambda_3^p}+1\right)\Gamma\left(2f_{nk}^s\right)}{\Gamma\left(T_{nk}^s+1\right)} \tag{47b}$$

$${}_3F_2\left(-n, X_{nk}^s, 2f_{nk}^s; 1+2\sqrt{\lambda_3^p}, T_{nk}^s+1; 1\right)$$

$$M_{3(nl^p m^p s^p)}^{p-ts} = N_{nk}^{np2} \frac{\Gamma\left(2\sqrt{\lambda_3^p}+2\right)\Gamma\left(2f_{nk}^p-1\right)}{\Gamma\left(T_{nk}^p+1\right)} \tag{47c}$$

$${}_3F_2\left(-n, X_{nk}^p, 2f_{nk}^p-1; 1+2\sqrt{\lambda_3^p}, T_{nk}^p+1; 1\right)$$

$$M_{4(nl^p m^p s^p)}^{p-ts} = N_{nk}^{np2} \frac{\Gamma\left(2\sqrt{\lambda_3^p}+1\right)\Gamma\left(2f_{nk}^p-2\right)}{\Gamma\left(T_{nk}^p-1\right)} \tag{47d}$$

$${}_3F_2\left(-n, X_{nk}^p, 2f_{nk}^p-2; 1+2\sqrt{\lambda_3^p}, T_{nk}^p-1; 1\right)$$

$$M_{5(nl^p m^p s^p)}^{p-ts} = N_{nk}^{np2} \frac{\Gamma\left(2\sqrt{\lambda_3^p}+1\right)\Gamma\left(2f_{nk}^s-1\right)}{\Gamma\left(T_{nk}^p\right)} \tag{47e}$$

$${}_3F_2\left(-n, X_{nk}^p, 2f_{nk}^p-1; 1+2\sqrt{\lambda_3^p}, T_{nk}^p; 1\right)$$

$$M_{6(nl^p m^p s^p)}^{p-ts} = N_{nk}^{np2} \frac{\Gamma\left(2\sqrt{\lambda_3^p}+2\right)\Gamma\left(2f_{nk}^p-2\right)}{\Gamma\left(T_{nk}^p\right)} \tag{47f}$$

$${}_3F_2\left(-n, X_{nk}^p, 2f_{nk}^p-2; 1+2\sqrt{\lambda_3^p}, T_{nk}^p; 1\right)$$

$$M_{7(nl^p m^p s^p)}^{p-ts} = N_{nk}^{np2} \frac{\Gamma\left(2\sqrt{\lambda_3^p}+3\right)\Gamma\left(2f_{nk}^p-2\right)}{\Gamma\left(T_{nk}^p+1\right)} \tag{47j}$$

$${}_3F_2\left(-n, X_{nk}^p, 2f_{nk}^p-2; 1+2\sqrt{\lambda_3^p}, T_{nk}^p+1; 1\right)$$

with X_{nk}^p and T_{nk}^p are equal $2\sqrt{\lambda_3^s}+2f_{nk}^p+n+1$ and $2\sqrt{\lambda_3^s}+2f_{nk}^p$, respectively.

D The corrected energy for the ITSP-ICHLTi in deformation Dirac theory symmetries

The main goal highlighted in this subsection is to find the contribution resulting from topological properties using our proper strategy that we have successfully applied in previous works and that we strive to develop in each new work. We can say that the global relativistic energy in the perspective of deformation Dirac theory is produced with the ITSP-ICHLTi model as a result of a major contribution to relativistic energy known in the literature under the TSP-ICHLTi model in the usual Dirac theory and which we paved the way for through a quick look at the spin(p-spin)-symmetry in Eqs. (17) and (18), while the new contribution is produced from the topological properties under space-space deformation, which can be evaluated through several contributions. We will examine three of them.

D1 The corrected spin-orbital energy for the ITSP-ICHLTi in deformation Dirac theory symmetries

The first is produced by the perturbed spin-orbit effective potentials ($\Xi_{ts}^{pert}(r)$ and $\Upsilon_{ts}^{pert}(r)$), which correspond to spin symmetry and pseudospin symmetry, respectively. These perturbed effective potentials are obtained by replacing the coupling of the angular momentum ($\mathbf{L}$, $\mathbf{L}^p$) operators and the NC vector Θ with the new equivalent couplings ($\Theta\mathbf{LS}$, $\Theta\mathbf{L}^p\mathbf{S}^p$) for spin-symmetry and p-spin-symmetry, respectively (with $\Theta^2=\Theta_{12}^2+\Theta_{23}^2+\Theta_{13}^2$). This degree of freedom comes into consideration when the infinitesimal NC vector Θ is arbitrary. We have oriented the two (spin-spin-s and spin-s^p) of the fermionic particles to become parallel to the vector Θ which interacted with improved Trigonometric scarf potential, including generalized (Coulomb-Hulthén)-like tensor interaction. We aligned the fermionic particles' two spin-s and spin-s^p to become parallel to the vector Θ, which interacted with a trigonometric scarf potential, including a generalized (Coulomb-Hulthén)-like tensor interaction. Furthermore, we replace the new spin-orbit couplings ($\Theta\mathbf{LS}$, $\Theta\mathbf{L}^p\mathbf{S}^p$) with the corresponding new physical forms $\Theta/2\mathbf{G}^2$ and $\Theta/2\mathbf{G}^{p2}$, respectively, with $\mathbf{G}^2=\mathbf{J}^2-\mathbf{L}^2-\mathbf{S}^2$ and $\mathbf{G}^{p2}=\mathbf{J}^2-\mathbf{L}^{p2}-\mathbf{S}^{p2}$ for a spin (p-spin)-symmetry, respectively. It is known, in RQM, the operators ($\mathbf{H}_{rnc}^{ts}$, $\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{L}^{p2}$, $\mathbf{S}^2$, $\mathbf{S}^{p2}$ and $\mathbf{J}_z$) form a complete set of conserved physics quantities, and the eigenvalues of the operators ($\mathbf{G}^2$ and $\mathbf{G}^{p2}$) are equal to the values

$$\begin{cases} F(j,l,s)=\left[j(j+1)-l(l+1)-3/4)\right]/2 \\ \text{with } \left|l-1/2\right|\le j\le \left|l+1/2\right| \\ F\left(j,l^p,s^p\right)=\left[j(j+1)-l^p(l^p-1)-3/4)\right]/2 \\ \text{with } \left|l^p-1/2\right|\le j\le \left|l^p+1/2\right| \end{cases}$$

for spin-symmetry and p-spin-symmetry, respectively. As a direct consequence, the partially corrected energies $\Delta E_{ts}^{so-s}(n,\alpha,V_0,H_c,H_H,\Theta,j,l,s)\equiv\Delta E_{ts}^{so-s}$ and $\Delta E_{ts}^{so-p}(n,\alpha,V_0,H_c,H_H,\Theta,j,l^p,s^p)\equiv\Delta E_{ts}^{so-sp}$ due to the perturbed effective potentials $\Xi_{ts}^{pert}(r)$ and $\Upsilon_{ts}^{pert}(r)$ produced for the $\left(n,l,l^p,m,m^p,s,s^p\right)^{th}$ excited state, in deformation Dirac theory symmetries as follows:

$$\begin{cases} \Delta E_{ts}^{so-s}=\Theta\left(j(j+1)-k(k+1)-s(s+1)\right) \\ \qquad\left\langle Z\right\rangle_{(nlms)}^{ts}\left(n,\alpha,V_0,H_c,H_H\right) \\ \Delta E_{ts}^{so-p}=\Theta\left(j(j+1)-k(k-1)-s^p(s^p+1)\right) \\ \qquad\left\langle Z^p\right\rangle_{(nlms)}^{ts}\left(n,\alpha,V_0,H_c,H_H\right) \end{cases} \tag{48}$$

The global two expectation values $\left\langle Z\right\rangle_{(nlms)}^{ts}\left(n,\alpha,V_0,H_c,H_H\right)$ and $\left\langle Z^p\right\rangle_{\left(nl^pm^ps^p\right)}^{ts}\left(n,\alpha,V_0,H_c,H_H\right)$ for a spin/(p-spin)-symmetry, respectively are determined from the following expressions:

$$\begin{cases} \left\langle Z\right\rangle_{(nlms)}^{ts}\left(n,\alpha,V_0,H_c,H_H\right)=\sum_{\gamma=1}^{7}T_{nk}^{\gamma s}M_{\gamma(nlms)}^{s-ts}, \\ \left\langle Z^p\right\rangle_{\left(nl^pm^ps^p\right)}^{ts}\left(n,\alpha,V_0,H_c,H_H\right)=\sum_{\gamma=1}^{7}T_{nk}^{\gamma p}M_{\gamma\left(nl^pm^ps^p\right)}^{p-ts}, \end{cases} \tag{49}$$

where $\left(\gamma=\overline{1,7}\right)$ while ($T_{nk}^{\gamma sp}$, $T_{nk}^{\gamma p}$) and ($M_{\gamma(nlms)}^{s-ts}$, $M_{\gamma\left(nl^pm^ps^p\right)}^{p-ts}$) are determined from Eq. (40) and (Eqs. (45) and (47)), respectively.

D2 The corrected magnetic energy for the ITSP-ICHLTi in the Dirac theory symmetries

The second main part is obtained from the magnetic effect of the perturbative effective potentials ($\Xi_{ts}^{pert}(r)$ and $\Upsilon_{ts}^{pert}(r)$) under the ITSP-ICHLTi model in the deformation of Dirac theory symmetries. These effective potentials are achieved when we replace both ($\mathbf{L}\Theta$, $\mathbf{L}^p\Theta$) by ($\eta\aleph\mathbf{L}_z$, $\eta\aleph\mathbf{L}_z^p$), respectively, and Θ_{12} by $\eta\aleph$, here ($\aleph$ and η) are present the intensity of the magnetic field induced by the effect of the deformation of space-space geometry and a new infinitesimal noncommutativity parameter, so that the physical unit of the original noncommutativity parameter $\left[\Theta_{12}\right]\equiv$ (length)^2 is the same unit of $\eta\aleph$, we have also need to apply

$$\begin{cases} \left\langle n',l',m',s'\middle|\mathbf{L}_z\middle|n,l,m,s\right\rangle= \\ m\delta_{m'm}\delta_{l'l}\delta_{n'n}\delta_{ss'}\text{ with }\left(-l\le m\le l\right) \\ \left\langle n',l^p,m^p,s^{p'}\middle|\mathbf{L}_z^p\middle|n,l^p,m^p,s^p\right\rangle= \\ m^p\delta_{m^pm^p}\delta_{l^{p'}l^p}\delta_{n'n}\delta_{s^ps^{p'}}\text{ with }\left(-l^p\le m^p\le l^p\right) \end{cases}$$

for a spin(p-spin)-symmetry, respectively. These data allow for the discovery of the new energy shift ΔE_{ts}^{mg-s}($n,\alpha,V_0,H_c,H_H,\eta,m$) $\equiv \Delta E_{ts}^{mg-s}$ and ΔE_{ts}^{mg-p}($n,\alpha,V_0,H_c,H_H,\eta,m^p$) $\equiv \Delta E_{ts}^{mg-p}$ due to the perturbed Zeeman effect created by the influence of the ITSP-ICHLTi model for the $\left(n,l,m,l^p,m^p,s,s^p\right)^{th}$ excited state in the deformation Dirac theory symmetries as follows:

$$\begin{cases} \Delta E_{ts}^{mg-s} = \eta\aleph\langle Z\rangle_{(nlms)}^{ts}\left(n,\alpha,V_0,H_c,H_H\right)m \\ \Delta E_{ts}^{mg-p} = \eta\aleph\left\langle Z^p\right\rangle_{\left(nl^pm^ps^p\right)}^{ts}\left(n,\alpha,V_0,H_c,H_H\right)m^p \end{cases} \tag{50}$$

D3 The corrected rotating energy for the ITSP-ICHLTi in the Dirac theory symmetries

We are about to find the third part, which is no less essential than the first two sections discussed the development of self-energy values as a result of deformation in space-space. As indicated in Eqs. (38) and (39), this physical phenomenon is caused by the influence of perturbed effective potentials ($\Xi_{ts}^{pert}(r)$, $\Upsilon_{ts}^{pert}(r)$). The fermionic particles are thought to be rotating at an angular velocity $\boldsymbol{\omega}$. By substituting for the arbitrary vector $\boldsymbol{\Theta}$, the characteristics of this subjective phenomenon can be identified $\chi\boldsymbol{\omega}$. This allows us to replace the two couplings ($\mathbf{L}\boldsymbol{\Theta}$, $\mathbf{L}^p\Theta$) for spin-symmetry and p-spin-symmetry, respectively, with ($\chi\mathbf{L}\boldsymbol{\omega}$, $\chi\mathbf{L}^p\boldsymbol{\omega}$) as follows:

$$\begin{pmatrix} \mathbf{L}\Theta \\ \mathbf{L}^p\Theta \end{pmatrix} \rightarrow \chi\begin{pmatrix} \mathbf{L}\boldsymbol{\omega} \\ \mathbf{L}^p\boldsymbol{\omega} \end{pmatrix}. \tag{51}$$

Here χ is just an infinitesimal real proportional constant. We can express the effective potentials ($\Sigma_{pert}^{ts-rot}(s)$, $\Delta_{pert}^{ts-rot}(s)$) which induced the rotational movements of the fermionic particles as follows:

$$\begin{pmatrix} \Xi_{ts}^{pert}(s) \\ \Upsilon_{ts}^{pert}(s) \end{pmatrix} \rightarrow \begin{pmatrix} \Sigma_{pert}^{ts-rot}(s) \\ \Delta_{pert}^{ts-rot}(s) \end{pmatrix} = \chi\begin{pmatrix} \left(\sum_{\gamma=1}^{7} T_{nk}^{\gamma s} M_{\gamma(nlms)}^{s-ts}\right)\mathbf{L}\boldsymbol{\omega} \\ \left(\sum_{\gamma=1}^{7} T_{nk}^{\gamma p} M_{\gamma\left(nl^pm^ps^p\right)}^{p-ts}\right)\mathbf{L}^p\boldsymbol{\omega} \end{pmatrix} \tag{52}$$

To simplify the calculations, we chose a rotational velocity $\boldsymbol{\omega}$ parallel to the (Oz) axis $\left(\boldsymbol{\omega} = \omega\mathbf{e}_z\right)$; this of course does not change the physical characteristics of the investigated problem as much as it simplifies the calculations. The spin-orbit couplings are then transformed into new physical phenomena as follows:

$$\begin{pmatrix} \Sigma_{pert}^{ts-rot}(s)\mathbf{L}\boldsymbol{\omega} \\ \Delta_{pert}^{ts-rot}(s)\mathbf{L}^p\boldsymbol{\omega} \end{pmatrix} = \chi\omega\begin{pmatrix} \Sigma_{pert}^{ts-rot}(s)\mathbf{L}_z \\ \Delta_{pert}^{ts-rot}(s)\mathbf{L}_z^p \end{pmatrix} \tag{53}$$

All of this physical information enables the discovery of the new corrected energies ΔE_{ts}^{rot-s}($n,\alpha,V_0,H_c,H_H\ \chi\ m$) and ΔE_{ts}^{rot-p}($n,\alpha,V_0,H_c,H_H,\chi,m^p$) due to the perturbed effective potentials ($\Sigma_{pert}^{ts-rot}(s)$ and $\Delta_{pert}^{ts-rot}(s)$) which are generated automatically by the influence of the improved Trigonometric scarf potential including generalized (Coulomb-Hulthén)-like tensor interaction for the $\left(n,l,j,m,l^p,m^p,s,s^p\right)^{th}$ excited state in DDT symmetries as follows:

$$\begin{pmatrix} \Delta E_{ts}^{rot-s} \\ \Delta E_{ts}^{rot-p} \end{pmatrix} = \chi\omega\begin{pmatrix} \langle Z\rangle_{(nlms)}^{ts} m \\ \left\langle Z^p\right\rangle_{\left(nl^pm^ps^p\right)}^{ts} m^p \end{pmatrix} \tag{54}$$

It is worth noting that the authors of ref. [89] investigated rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gases in two and three-dimensional space at zero temperature, but in this case, the rotational term was added to the Hamiltonian operator, whereas in our case, the two rotation operators ($\Sigma_{pert}^{ts-rot}(s)\mathbf{L}\boldsymbol{\omega}$ and $\Delta_{pert}^{ts-rot}(s)\mathbf{L}^p\boldsymbol{\omega}$) appear automatically due to the augmented symmetries resulting from the deformation of space-space under the improved Trigonometric scarf potential including generalized (Coulomb-Hulthén)-like tensor interaction. For fermionic particles/anti-particles, the eigenvalues ($F(j,l,s)$ and $F\left(j,l^p,s^p\right)$) of the operations ($\mathbf{G}^2$ and $\mathbf{G}^{p2}$) are equal to the following values:

$$\begin{cases} F(j,l,s) = \left[j(j+1) - l(l+1) - 3/4\right]/2 \\ F\left(j,l^p,s^p\right) = \left[j(j+1) - l^p(l^p+1) - 3/4\right]/2 \end{cases}$$

respectively. Thus, for the case of spin-1/2 fields, the possible values of j are ($l\pm 1/2$ and $l^p \pm 1/2$) for spin symmetry $F(j,l,s)$ and pseudospin symmetry $F\left(j,l^p,s^p\right)$, as follows:

$$F(j = l\pm 1/2, s = 1/2) = \frac{1}{2}\begin{cases} l & \text{Up polarity}: \ j = l+1/2 \\ -(l+1) & \text{Down polarity}: \ j = l-1/2 \end{cases} \tag{55}$$

and

$$F\left(j = l^p \pm 1/2, s^p = 1/2\right) = \frac{1}{2}\begin{cases} l^p & \text{Up polarity}: \ j = l^p+1/2 \\ -\left(l^p+1\right) & \text{Down polarity}: \ j = l^p-1/2 \end{cases} \tag{56}$$

The global relativistic energy E_{nc}^{ts-s}($n,\alpha,V_0,H_c,H_H,\Theta,\eta$ $,\chi,j,\ l,s,m$) (E_{nc}^{ts-s}, in short) and E_{nc}^{ts-p}(n,α,V_0,H_c,

$H_H, \Theta, \eta, \chi, j,\ l^p, s^p, m^p$) ($E_{nc}^{ts-s}$, in short)for spin-1/2 with improved Trigonometric scarf potential, including a generalized (Coulomb-Hulthén)-like tensor interaction, in the DDT symmetries, corresponding to the generalized $\left(n,l,m,s,l^p,m^p,s^p\right)^{th}$ excited states:

$$E_{nc}^{ts-s} = E_{nk}^{s} + \langle Z\rangle_{(nlms)}^{ts}\left(n,\alpha,V_0,H_c,H_H\right)\times \begin{bmatrix} (\eta\aleph+\chi\omega)m+\frac{\Theta}{2} \\ \begin{cases} l & \text{for UP}: \ j=l+1/2 \\ -(l+1) & \text{for DP}: j=l-1/2 \end{cases}\end{bmatrix} \tag{57a}$$

and

$$E_{nc}^{ts-p} = E_{nk}^{p} + \left\langle Z^p\right\rangle_{\left(nl^p m^p s^p\right)}^{ts}\left(n,\alpha,V_0,H_c,H_H\right)\times \begin{bmatrix} (\eta\aleph+\chi\omega)m^p+\frac{\Theta}{2} \\ \begin{cases} l^p & \text{for UP}: \ j=l^p+1/2 \\ -\left(l^p+1\right) & \text{for DP}: j=l^p-1/2 \end{cases}\end{bmatrix} \tag{57b}$$

Where E_{nk}^{s} and E_{nk}^{p} are usual relativistic energies under trigonometric scarf potential including a (Coulomb-Hulthén)-like tensor interaction obtained from equations of energy in Eqs. (17) and (18). here UP and DP are Up polarity and Down polarity, respectively. We can now generalize our obtained energies E_{g-nc}^{ts-s} and E_{g-nc}^{ts-p} which were produced with the globally induced two potentials $\Sigma_{t\_ts}^{pert}(r)$ and $\Delta_{t\_ts}^{pert}(r)$ for a spin and p-spin symmetries corresponding (UC) and (LC) ($F_{nk}^{s}(s)$ and $G_{nk}^{s}(s)$) and ($F_{nk}^{p}(s)$ and $G_{nk}^{p}(s)$), respectively as:

$$E_{g-nc}^{ts-s} = E_{nc}^{ts-s}\theta\left(\left|E_{nc}^{ts-s}\right|\right) - E_{nc}^{ts-s}\theta\left(-\left|E_{nc}^{ts-s}\right|\right) = \begin{cases} E_{nc}^{ts-s} & \text{for UC of spin symmetry} \\ -E_{nc}^{ts-s} & \text{for LC of spin symmetry} \end{cases} \tag{58a}$$

and

$$E_{g-nc}^{ts-p} = E_{nc}^{ts-p}\theta\left(\left|E_{nc}^{ts-p}\right|\right) - E_{nc}^{ts-p}\theta\left(-\left|E_{nc}^{ts-p}\right|\right) = \begin{cases} E_{nc}^{ts-p} & \text{for UC of p-spin symmetry} \\ -E_{nc}^{ts-p} & \text{for LC of p-spin symmetry} \end{cases} \tag{58b}$$

VI. THE IMPROVED TRIGONOMETRIC SCARF POTENTIAL PROBLEM IN D-NRQM SYMMETRIES:

To achieve a nonrelativistic study of the improved trigonometric scarf potential, we will apply the principle of the nonrelativistic limit, in deformation nonrelativistic quantum mechanics (D-NRQM) symmetries through two stages. The first step corresponds to the nonrelativistic limit, in usual nonrelativistic quantum energy. This is done by applying the following steps, we replace: $\left(H_c,H_H,C_s\right)\rightarrow(0,0,0)$, $E_{nk}^{s}+M\rightarrow 2\mu$, $E_{nk}^{s}-M\rightarrow E_{nl}^{nr}$ and $k(k+1)\rightarrow l(l+1)$. This allows us to obtain the nonrelativistic energy levels as:

$$E_{nl}^{nr} = \frac{2\alpha^2}{\mu}\left[\frac{\frac{2\mu V_0}{\alpha^2} - l(l+1) - n\left(n+1+\frac{1}{2n}\right) - \left(n+\frac{1}{2}\right)\sqrt{(2l+1)^2-\frac{8\mu V_0}{\alpha^2}}}{2n+1+\sqrt{(2l+1)^2-\frac{8\mu V_0}{\alpha^2}}}\right]^2 \tag{59}$$

Now, the second step corresponds to the transformation of the relativistic coefficients $T_{nk}^{\gamma s}$ $\left(\gamma=\overline{1,7}\right)$ under the previous correspondence to the new nonrelativistic coefficient:

$$T_{nk}^{nr} = 2\left(l(l+1) - \frac{V_0\aleph_0}{\alpha^2}\right)\alpha\,. \tag{60}$$

This allows us to re-export the relativistic expectation values of spin symmetry $\langle Z\rangle_{(nlms)}^{ts}(n,\alpha,V_0,H_c,H_H)$ in Eq. (49) from the corresponding nonrelativistic expectation values $\langle Z\rangle_{(nlms)}^{ts-nr}\left(n,\alpha,V_0\right)$ as:

$$\langle Z\rangle_{(nlms)}^{ts-nr}\left(n,\alpha,V_0\right) = 2\alpha\left(l(l+1) - \frac{V_0\aleph_0}{\alpha^2}\right)M_{1(nlms)}^{sp-ts}, \tag{61}$$

which allows us to express the nonrelativistic correction energy ΔE_{nc-nr}^{ts} ($n,\alpha,V_0,H_c,H_H,\Theta,\eta,\chi,j,\ l,s,m$) produced by the improved Trigonometric scarf potential problems as

$$\Delta E_{nc-nr}^{ts}\left(n,\alpha,V_0,\Theta,\eta,\chi,j,l,s,m\right) = \langle Z\rangle_{(nlms)}^{ts-nr}\left(n,\alpha,V_0\right)\left[(\eta\aleph+\chi\boldsymbol{\omega})m + \frac{\Theta}{2}\begin{cases} l & \text{UP}: j=l+1/2 \\ -(l+1) & \text{DP}: j=l-1/2 \end{cases}\right] \tag{62}$$

According to the time-independent perturbation theory, in D-NRQM symmetries, considering that the improved Trigonometric scarf potential is combined from the main part (Trigonometric scarf potential in NRQM symmetries) and a perturbed part that can be considered an infinitesimal term compared to the principal term, thus, the global nonrelativistic energy $E_{nc-nr}^{ts}(n,\alpha,V_0,\Theta,\eta,\chi,j,\ l,s,m)$ is the sum of usual energy E_{nl}^{ts} in Eq. (59), and the obtained infinitesimal correction ΔE_{nc-nr}^{ts}, in Eq. (62), as follows:

$$E_{nc-nr}^{ts} = \frac{2\alpha^2}{\mu}\left[\frac{\frac{2\mu V_0}{\alpha^2} - l(l+1) - n\left(n+1+\frac{1}{2n}\right) - \left(n+\frac{1}{2}\right)\sqrt{(2l+1)^2 - \frac{8\mu V_0}{\alpha^2}}}{2n+1+\sqrt{(2l+1)^2 - \frac{8\mu V_0}{\alpha^2}}}\right]^2 \qquad (63)$$
$$+\langle Z\rangle_{(nlms)}^{ts-nr}\left[(\eta\aleph+\chi\mathbf{\varpi})m + \frac{\Theta}{2}\begin{cases} l & \text{for UP} \\ -(l+1) & \text{for DP}\end{cases}\right].$$

It should be noted that the corrected energy ΔE_{nc-nr}^{ts} expressed in Eq. (62) is due to the effect of the perturbed potential $V_{pert}^{ts}(r)$:

$$V_{pert}^{ts}(r) = \left(\frac{l(l+1)}{r^4} - \frac{1}{2r}\frac{\partial V_{ts}(r)}{\partial r}\right)\mathbf{L}\Theta + O(\Theta^2). \qquad (64)$$

The first term in Eq. (64) is due to the centrifuge term $l(l+1)r_{nc}^{-2}$ in D-NRQM symmetries, which equals the usual centrifuge term $l(l+1)r^{-2}$ plus the perturbative centrifuge term $l(l+1)r^{-4}\mathbf{L\Theta}$ while the second term is produced with the effect of the improved Trigonometric scarf potential. This is one of the most important new results of this research. It is worth noting that for the three-simultaneous limits $(\Theta,\eta,\chi)\to(0,0,0)$, we recover the energy equations for the spin symmetry and the p-spin symmetry under the Trigonometric scarf potential, including a generalized (Coulomb-Hulthén)-like tensor interaction, which is presented in the ref. [7].

VI. CONCLUSIONS

In summary, this work presents an approximate analytical solution of the 3-dimensional deformed Dirac equation with the improved Trigonometric scarf potential within the generalized (Hulthén and Coulomb) like tensor interaction under pseudospin and spin symmetry limits with an arbitrary spin-orbit coupling quantum number k. To do so, we have dealt with the centrifugal potential term using the Greene-Aldrich approximation. To do so, we have dealt with the centrifugal potential term using the Greene-Aldrich approximation. We obtained new approximate bound-state energies that appear to be sensitive to the quantum numbers (j,k,l,m,l^p,m^p,s,s^p), potential depths (V_0,H_c,H_H) of the studied potentials, potential range α, and noncommutativity parameters (Θ,η,χ) under the condition of spin and pseudospin symmetry. We also ended our research with this treatment of the nonrelativistic limit of the improved Trigonometric scarf potential in D-NRQM symmetries. It is worth mentioning that for all cases, to achieve the three simultaneous limits $(\Theta,\eta,\chi)\to(0,0,0)$, the ordinary physical quantities are recovered in ref. [7]. Finally, a feature of a noncommutative geometry on the 3-dimensional deformed Dirac equation with the improved Trigonometric scarf potential within the generalized (Hulthén and Coulomb) like tensor interaction would be the presence of many physics phonemes, such as spin-orbit and pseudospin-orbit, modified Zeeman effect, and others, which cause the behavior of topological properties of deformed space-space. However, when compared to related work in the literature, our new results revealed a great improvement.

ACKNOWLEDGEMENTS

This work has been partly supported by the Algerian Ministry of Higher Education and Scientific Research and DGRST and by the Laboratory of Physics and Material Chemistry in the university of M'sila. The author is grateful to the reviewers

REFERENCES

[1] Wei, G.-F., Duan, X.-Y., and Liu, X.-Y., *Algebraic approach to spin symmetry for the Dirac equation with the Trigonometric scarf potential*, Int. J. Mod. Phys. A **25**(08), 1649-1659 (2010). https://doi.org/10.1142/s0217751x10048287

[2] Cooper, F., Khare, A., and Sukhatme, U., *Supersymmetry and quantum mechanics*, Physics Reports **251**(5-6), 267-385 (1995). https://doi.org/10.1016/0370-1573(94)00080-m

[3] Alvarez-Castillo, D.E., and Kirchbach, M., *Exact spectrum and wave functions of the hyperbolic scarf potential in terms of finite Romanovski polynomials*, Rev. Mex. Fis. E **53**(2), 143-154 (2007).

[4] Lévai, G., *On the pseudo-norm and admissible solutions of the -symmetric scarf I potential*, J. Phys. A: Math. Gen. **39**(32), 10161-10169 (2006). https://doi.org/10.1088/0305-4470/39/32/s17

[5] Quesne, C., *Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry*, J. Phys. A: Math. Theor. **41**(39), 392001 (2008). https://doi.org/10.1088/1751-8113/41/39/392001

[6] Falaye, B. J., and Oyewumi, K. J., *Solutions of the Dirac Equation with Spin and Pseudospin Symmetry for the Trigonometric scarf Potential in D-dimensions*, Afr. Rev. Phys. **6**(0025), 211-220 (2011).

[7] Onate, C. A., Adebimpe, O., Lukman, A. F., Adama, I. J., Davids, E. O., and Dopamu, K. O., *Approximate solutions of the Dirac equation with Coulomb-Hulthén-like tensor interaction*, Results in Physics **11**, 1094-1099 (2018). https://doi.org/10.1016/j.rinp.2018.10.052

[8] Mezincescu, L., Star operation in quantum mechanics, https://arxiv.org/abs/hep-th/0007046 (2000).

[9] Gouba, L., *A comparative review of four formulations of noncommutative quantum mechanics*, Inter. J. Mod. Phys. A **31**(19), 1630025 (2016). https://doi.org/10.1142/s0217751x16300258

[10] Bopp, F., *La mécanique quantique est-elle une mécanique statistique classique particulière*, Ann. Inst. Henri Poincaré **15**, 81-112 (1956). http://www.numdam.org/item/AIHP_1956__15_2_81_0.pdf

[11] Gamboa, J., Loewe, M., and Rojas, J.C., *Noncommutative quantum mechanics*, Phys. Rev. D **64**(6), 067901 (2001). https://doi.org/10.1103/physrevd.64.067901

[12] Birrell, N. D., Birrell, N. D., and Davies, P. C. W., *Quantum fields in curved space* (Cambridge University Press, Cambridge) 1982.

[13] Ho, P.-M., and Kao, H.-C., *Noncommutative quantum mechanics from noncommutative quantum field theory*, Phys. Rev. Lett. **88**(15), 151602-1 (2002). https://doi.org/10.1103/physrevlett.88.151602

[14] Connes, A., Douglas, M.R., and Schwarz, A., *Noncommutative geometry and matrix theory*, JHEP **1998**(02), 003-003 (1998).

https://doi.org/10.1088/1126-6708/1998/02/003
[15] Mirza, A., and Mohadesi, M., *The Klein-Gordon and the Dirac oscillators in a noncommutative space*, Commun. Theor. Phys. **42**(5), 664-668 (2004). https://doi.org/10.1088/0253-6102/42/5/664
[16] Capozziello, S., Lambiase, G., and Scarpetta, G., *Generalized Uncertainty Principle from Quantum Geometry*, Int. J. Theor. Phys. **39**, 15-22 (2000). https://doi.org/10.1023/A:1003634814685
[17] Doplicher, S., Fredenhagen, K., and Roberts, J.E., *Spacetime quantization induced by classical gravity*, Phys. Lett. B **331**(1-2), 39-42 (1994). https://doi.org/10.1016/0370-2693(94)90940-7
[18] Witten, E., *Reflections on the Fate of Spacetime*, Phys. Today **49**(4), 24-30 (1996). https://doi.org/10.1063/1.881493
[19] Kempf, A., Mangano, G., and Mann, R.B., *Hilbert space representation of the initial length uncertainty relation*, Phys. Rev. D **52**(2), 1108 (1995). https://doi.org/10.1103/physrevd.52.1108.
[20] Adler, R.J., and Santigo, D.I., *On gravity and the uncertainty principal*, Mod. Phys. Lett. A **14** (20), 1371-1381 (1999). https://doi.org/10.1142/s0217732399001462.
[21] Kanazawa, T., Lambiase, G., Vilasi, G., and Yoshioka, A., *Noncommutative schwarzschild geometry and generalized uncertainty principle*, Eur. Phys. J. C. **79**(2) (2019). https://doi.org/10.1140/epjc/s10052-019-6610-1
[22] Scardigli, F., *Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment*, Phys. Lett. B **452**(1-2), 39-44 (1999). https://doi.org/10.1016/s0370-2693(99)00167-7
[23] Snyder, H. S., *Quantized space-time*, Phys. Rev. **71**, 38-41 (1947). https://doi.org/10.1103/PhysRev.71.38; *The electromagnetic field in quantized space-time*, Phys. Rev.**72**, 68 (1947). https://doi.org/10.1103/PhysRev.72.68
[24] Connes, A., *Noncommutative geometry* (ISBN-9780121858605) (1994).
[25] Connes, A., and Lott, J., *Particle models and noncommutative geometry*, Nucl. Phys. B. **18**, 29-47 (1991).
[26] Seiberg, N., and Witten, E., *String theory and noncommutative geometry*, JHEP **1999**(09), 032-032 (1999). https://doi.org/10.1088/1126-6708/1999/09/032
[27] Nicolini, P., *Noncommutative black holes, the final appeal to quantum gravity: A review*, Int. J Mod. Phys. A **24**(07), 1229-1308 (2009). https://doi.org/10.1142/s0217751x09043353
[28] Maireche, A., *A theoretical study of the modified equal scalar and vector Manning-Rosen potential within the deformed Klein-Gordon and Schrödinger in RNCQM and NRNCQM symmetries*, Rev. Mex. Fis. **67**(5), 050702 1-18 (2021). https://doi.org/10.31349/RevMexFis.67.050702
[29] Maireche, A., *A new theoretical investigation of the modified equal scalar and vector Manning-Rosen plus quadratic Yukawa potential within the deformed Klein--Gordon and Schrödinger equations using the improved approximation of the centrifugal term and Bopp's Shift Method in RNCQM and NRNCQM symmetries*, SPIN Journal **11**, No. 04, 2150029 (2021). https://doi.org/10.1142/S2010324721500296
[30] Maireche, A., *New bound-state solutions of the deformed Klien-Gordon and Schrödinger equations for arbitrary l-state with modified equal vector and scalar Manning-Rosen plus a class of Yukawa potentials in RNCQM and NRNCQM symmetries*, J. Phys. Stud. **25**(4), 4301 (2021). https://doi.org/10.30970/jps.25.4301
[31] Maireche, A., *The investigation of approximate solutions of deformed Klein-Fock-Gordon and Schrödinger equations under modified Equal Scalar and Vector Manning-Rosen and Yukawa Potentials by using the Improved Approximation of the Centrifugal term and Bopp's shift Method in NCQM Symmetries*, Lat. Am. J. Phys. Educ. **15**, No. 2, 2310-1 (2021).
[32] Maireche, A., *Investigations on the relativistic interactions in one-electron atoms with modified Yukawa potential for spin 1/2 particles*, Int. Front. Sci. Lett. **11**, 29 (2017). https://doi.org/10.18052/www.scipress.com/IFSL.11.29
[33] Maireche, A., *Nonrelativistic treatment of hydrogen-like and neutral atoms subjected to the generalized perturbed Yukawa potential with centrifugal barrier in the symmetries of noncommutative quantum mechanics*, Int. J. Geo. Met. Mod. Phys. **17**(5), 2050067 (2020). https://doi.org/10.1142/S021988782050067X
[34] Yi, Y., Kang, K., Jian-Hua, W., and Chi-Yi, C., *Spin-1/2 relativistic particle in a magnetic field in NC phase space*, Chin. Phys. C **34**(5), 543-547 (2010). https://doi.org/10.1088/1674-1137/34/5/005
[35] Maireche, A., *A theoretical model of deformed Klein--Gordon equation with generalized modified screened Coulomb plus inversely quadratic Yukawa potential in RNCQM Symmetries*, Few-Body Syst. **62**, 12 (2021). https://doi.org/10.1007/s00601-021-01596-2
[36] Maireche, A., *Modified unequal mixture scalar vector Hulthén--Yukawa potentials model as a quark-antiquark interaction and neutral atoms via relativistic treatment using the improved approximation of the centrifugal term and Bopp's Shift method*, Few-Body Syst. **61**, 30 (2020). https://doi.org/10.1007/s00601-020-01559-z.
[37] Maireche, A., *Heavy quarkonium systems for the deformed unequal scalar and vector Coulomb--Hulthén potential within the deformed effective mass Klein--Gordon equation using the improved approximation of the centrifugal term and Bopp's shift method in RNCQM symmetries*, Int. J. Geo. Met. Mod. Phys. **18**(13), 2150214 (2021). https://doi.org/10.1142/S0219887821502145
[38] Maireche, A., *Bound-state Solutions of Klein-Gordon and Schrödinger Equations for Arbitrary* l*-state with Linear Combination of Hulthén and Kratzer Potentials*, Afr. Rev. Phys. **15**:003, 19-31 (2020).
[39] Samary, D.O., Guedezounme, S.L., and Kanfon, A.D., *Lattice Oscillator model on noncommutative space: Eigenvalues problem for the perturbation theory*, Braz. J. Phys. **49**, 458-470 (2019). https://doi.org/10.1007/s13538-019-00655-8
[40] Maireche, A., *A Model of Modified Klein--Gordon Equation with Modified Scalar-vector Yukawa Potential*, Afr. Rev Phys. 15: 0001, 1-11 (2020).
[41] Vacaru, S.I., *Exact solutions with noncommutative symmetries in Einstein and gauge gravity*, J. Math. Phys. **46**(4), 042503 (2005). https://doi.org/10.1063/1.1869538
[42] Maireche, A., *A New Approach to the Approximate Analytic Solution of the Three-Dimensional Schrödinger Equation for Hydrogenic and Neutral Atoms in the Generalized Hellmann Potential Model*, Ukr. J. Phys. **65**(11), 987-1001 (2020). https://doi.org/10.15407/ujpe65.11.987
[43] Bertolami, O., Rosa, J.G., De Aragao, C. M. L. Castorina, P., and Zappala, D., *Scaling of variables and the relation between noncommutative parameters in noncommutative quantum mechanics*, Mod. Phys. Lett. A **21**(10), 795-802 (2006). https://doi.org/10.1142/s0217732306019840
[44] N'Dolo, E. E., Samary, D. O., Ezinvi, B., and Hounkonnou, M.N., *Noncommutative Dirac and Klein--Gordon oscillators in the background of cosmic string: Spectrum and dynamics*, Int. J. Geo. Met. Mod. Phys. 17(05), 2050078 (2020). https://doi.org/10.1142/s0219887820500784
[45] Gnatenko, K. P., and Tkachuk, V. M., *Composite system in rotationally invariant noncommutative phase space*, Int. J. Mod. Phys. A **33**(07), 1850037 (2018). https://doi.org/10.1142/s0217751x18500379
[46] Maireche, A., *Bound-state solutions of the modified Klien-Gordon and Schrödinger equations for arbitrary l-state with the modified Morse potential in symmetries of noncommutative quantum mechanics*, J. Phys. Stud. **25**(1), 1002 (2021). https://doi.org/10.30970/jps.25.1002
[47] Gnatenko, K. P., *Composite system in noncommutative space and the equivalence principle*, Phys. Lett. A **377**(43), 3061-3066 (2013). https://doi.org/10.1016/j.physleta.2013.09.036
[48] Djemaïn, E. F., and Smail, H., *On quantum mechanics on noncommutative quantum phase space*, Commun. Theor. Phys. **41**(6),

837-844 (2004). https://doi.org/10.1088/0253-6102/41/6/837

[49] Valencia-Ortega, G., and Arias-Hernandez, L.-A., *Thermodynamic properties of diatomic molecule systems under SO 2,1) anharmonic Eckart potential*, International Journal of Quantum Chemistry **11**8(14), e25589 (2018). https://doi.org/10.1002/qua.25589

[50] Bertolami, O., Rosa, J. G., De Aragão, C. M. L., Castorina, P., and Zappalà, D., *Noncommutative gravitational quantum well*, Phys. Rev. D **72**(2) (2005) 025010. https://doi.org/10.1103/physrevd.72.025010

[51] Zhang, J., *Fractional angular momentum in non-commutative spaces*, Phys. Lett. B **584**(1-2), 204-209 (2004). https://doi.org/10.1016/j.physletb.2004.01.049

[52] Chaichian, M., Sheikh-Jabbari, M. M., and Tureanu, A., *Hydrogen atom spectrum and the Lamb Shift in noncommutative QED*, Phys. Rev. Lett. **86**(13), 2716-2719 (2001). https://doi.org/10.1103/physrevlett.86.2716

[53] Abreu, E. M. C., Neves, C., and Oliveira, W., *Noncommutativity from the symplection the point of view*, Int. J. Mod. Phys. A **21**, 5359-5369 (2006). https://doi.org/10.1142/s0217751x06034094

[54] Abreu, E. M. C., Analas neto, J., Mende, A. C. R., Neves, C., Oliveira, W., and Marcial, M. V., *Lagrangian formulation for noncommutative nonlinear systems*, Int. J. Mod. Phys. A **27**(09), 1250053 (2012). https://doi.org/10.1142/s0217751x12500534

[55] Wang, J., and Li, K., *The HMW effect in noncommutative quantum mechanics*, J. Phys. A: Math. Theor. **40**(9), 2197-2202 (2007). https://doi.org/10.1088/1751-8113/40/9/021

[56] Aghababae, S., and Rezaei, G., *Energy level splitting of a 2D hydrogen atom with Rashba coupling in non-commutative space*, Commun. Theor. Phys. **72**, 125101 (2020). https://doi.org/10.1088/1572-9494/abb7cc

[57] Maireche, A., *New relativistic bound states for modified pseudoharmonic potential of Dirac equation with spin and pseudo-spin symmetry in one-electron atoms*, Afr. Rev Phys. **12**, 0018, 130-143 (2017).

[58] Maireche, A., *New relativistic atomic mass spectra of quark (u, d and s) for extended modified Cornell potential in Nano and Plank's scales*, J. Nano- Electron. Phys. **8** (1), 01020 (2016). https://doi.org/10.21272/jnep.8(1).01020

[59] Maireche, A., *A new study of energy levels of hydrogenic atoms and some molecules for new more general exponential screened coulomb potential*, Open. Acc. J. Math. Theor. Phy. **1**(6), 232-237 (2018). https://doi.org/10.15406/oajmtp.2018.01.00040

[60] Maireche, A., *Solutions of Klein-Gordon equation for the modified central complex potential in the symmetries of noncommutative quantum mechanics*, Sri Lankan J. of Phys. **22**(1), 1-19 (2021). http://doi.org/10.4038/sljp.v22i1.8079

[61] Maireche, A., *The relativistic and nonrelativistic solutions for the modified unequal mixture of scalar and time-like vector Cornell potentials in the symmetries of noncommutative quantum mechanics*, Jordan J. Phys. **14** (1), 59-70 (2021). https://doi.org/10.47011/14.1.6

[62] Maireche, A., *A theoretical investigation of nonrelativistic bound state solution at finite temperature using the sum of modified Cornell plus inverse quadratic potential*, Sri Lankan J. of Phys. **21**, 11-36 (2020). http://doi.org/10.4038/sljp.v21i1.8069

[63] Maireche, A., *Extended of the Schrödinger equation with New Coulomb potentials plus linear and Harmonic radial terms in the symmetries of noncommutative quantum mechanics*, J. Nano-Electron. Phys. **10**(6), 06015-1 (2018). https://doi.org/10.21272/jnep.10(6).06015

[64] Maireche, A., *Heavy-light mesons in the symmetries of extended nonrelativistic quark model*, Yanbu J. Eng. Sci. **17**(1), 51-63 (2019). https://doi.org/10.53370/001c.23732

[65] Maireche, A., *Effects of two-dimensional noncommutative theories on bound states Schrödinger diatomic molecules under new modified Kratzer-type interactions*, ILCPA **76**, 1 (2017). https://doi.org/10.18052/www.scipress.com/ILCPA.76.1

[66] Motavalli, H., and Akbarieh, A. R., *Klein-Gordon equation for the Coulomb potential in noncommutative space*, Mod. Phys. Lett. A **25**(29), 2523-2528 (2010). https://doi.org/10.1142/s0217732310033529

[67] Maireche, A., *The Impact of deformed space-phase parameters into HAs and HLM systems with the improved Hulthen Plus Hellmann potentials model in the presence of temperature-dependent confined Coulomb potential within the framework of DSE*, Rev. Mex. Fis. **68**, no.5, 050702 1 (2022). https://doi.org/10.31349/RevMexFis.68.050702.

[68] Maireche, A., *The Klein--Gordon equation with modified Coulomb plus inverse-square potential in the noncommutative three-dimensional space*, Mod. Phys. Lett. A. **35**(5), 052050015 (2020). https://doi.org/10.1142/s0217732320500157

[69] Maireche, A., *Analytical expressions to energy eigenvalues of the hydrogenic atoms and the heavy light mesons in the framework of 3D-NCPS symmetries using the generalized Bopp's Shift method*, Bulg. J. Phys. **49** no.3, 239-265 (2022). https://doi.org/10.55318/bgjp.2022.49.3.239

[70] Maireche, A., *The Impact of deformed space-phase parameters into HAs and HLM systems with the improved Hulthen plus Hellmann potentials model in the presence of temperature-dependent confined Coulomb potential within the framework of DSE,* Rev. Mex. Fís. **68**(5), 050702-1 (2022). https://doi.org/10.31349/RevMexFis.68.050702

[71] Maireche, A., *Relativistic symmetries of bosonic particles and antiparticles in the background of the position-dependent mass for the improved deformed Hulthen plus deformed type-hyperbolic potential in 3D-EQM Symmetries*, East European Journal of Physics **2022**(4), 200-212 (2022). https://doi.org/10.26565/2312-4334-2022-4-21

[72] Maireche, A., *A new theoretical study of the deformed unequal scalar and vector Hellmann plus modified Kratzer potentials within the deformed Klein--Gordon equation in RNCQM symmetries,* Mod. Phys. Lett. A **36** (33), 2150232 (2021). https://doi.org/10.1142/S0217732321502321

[73] Darroodi, M., Mehraban, H., and Hassanabadi, S., *The Klein--Gordon equation with the Kratzer potential in the noncommutative space*, Mod. Phys. Let. A **33**(35), 1850203 (2018). https://doi.org/10.1142/s0217732318502036

[74] Maireche, A., *Theoretical investigation of the modified screened cosine Kratzer potential via relativistic and nonrelativistic treatment in the NCQM symmetries*, Lat. Am. J. Phys. Educ. **14** (3), 3310-1 (2020).

[75] Maireche, A., *New relativistic and nonrelativistic model of diatomic molecules and fermionic particles interacting with improved modified Mobius potential in the framework of noncommutative quantum mechanics symmetries*, Yanbu J. Eng. Sci. **18**(1), 10-32 (2021). https://doi.org/10.53370/001c.28090

[76] Maireche, A., *A New relativistic study for interactions in one-electron atoms (spin ½ particles) with modified Mie-type potential*, J. Nano-Electron. Phys. **8**(4), 04027 (2016). https://doi.org/10.21272/jnep.8(4(1)).04027

[77] Maireche, A., *Approximate arbitrary k state solutions of Dirac equation with improved inversely quadratic Yukawa potential within improved Coulomb-like tensor interaction in deformation quantum mechanics symmetries*, Few-Body Syst. **63**, 54 (2022). https://doi.org/10.1007/s00601-022-01755-z

[78] Maireche, A., *Relativistic symmetries of the deformed Dirac equation through the improved Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction in deformation quantum mechanics*, J. Phys. Stud. **26** No. 2, 2001(20 p) (2022). https://doi.org/10.30970/jps.26.2001

[79] Maireche, A., *On the interaction of an improved Schiöberg potential within the Yukawa tensor interaction under the background of deformed Dirac and Schrödinger equations*, Ind. J. Phys. **96**, 10

(2022). https://doi.org/10.1007/s12648-022-02433-w

[80] Maireche, A., *Deformed Dirac and Schrödinger equations with improved Mie-type potential for diatomic molecules and fermionic particles in the framework of extended quantum mechanics symmetries*, Ukr. J. Phys. **67**(7), 485-503 (2022). https://doi.org/10.15407/ujpe67.7.485

[81] Aounallah, H., and Boumali, A., Solutions of the Duffin--Kemmer equation in non-*commutative space of cosmic string and magnetic monopole with allowance for the Aharonov--Bohm and Coulomb potentials*, Phys. Part. Nuclei Lett. **16**, 195-205 (2019). https://doi.org/10.1134/S1547477119030038

[82] Saidi, A., and Sedra, M. B., *Spin-one (1+3)-dimensional DKP equation with modified Kratzer potential in the non-commutative space*, Mod. Phys. Lett. A **35**(5), 2050014 (2020). https://doi.org/10.1142/s0217732320500145

[83] Greene, R. L., and Aldrich, C., *Variational wave functions for a screened Coulomb potential*, Phys. Rev. A **14**(6), 2363-2366 (1976). https://doi.org/10.1103/physreva.14.2363

[84] Ahmadov, A.I., Demirci, M., Mustamin, M.F., Aslanova, S. M., and Sh. Orujova, M., *Analytical bound state solutions of the Dirac equation with the Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction*, The Eur. Phys. J. Plus **136**, 208 (2021). https://doi.org/10.1140/epjp/s13360-021-01163-y

[85] Ahmadov, A. I., Aslanova, S. M., Orujova, M. Sh., Badalov, S.V., and Dong, S.-H., *Approximate bound state solutions of the Klein-Gordon equation with the linear combination of Hulthén and Yukawa potentials*, Phys. Letter. A **383**, 3010 (2019). https://doi.org/10.1016/j.physleta.2019.06.043

[86] Ahmadov, A. I., Naeem, M., Qocayeva, M.V., and Tarverdiyeva, V.A., *Analytical Bound State Solutions of the Schrödinger Equation for the Manning - Rosen plus Hulthén potential within SUSY quantum mechanics*, Int. J. Mod. Phys. A **33**(03), 1850021 (2018). https://doi.org/10.1142/S0217751X18500215

[87] Dong, S.-H., Qiang, W.-C., Sun, G.-H., and Bezerra, V. B., *Analytical approximations to the l-wave solutions of the Schrödinger equation with the Eckart potential*, Phys. A: Math. Theor. **40**(34), 10535-10540 (2007). https://doi.org/10.1088/1751-8113/40/34/010

[88] Zhang, Y., *Approximate analytical solutions of the Klein-Gordon equation with scalar and vector Eckart potentials*, Phys. Scr. **78**(1), 015006 (2008). https://doi.org/10.1088/0031-8949/78/01/015006

[89] Bencheikh, K., Medjedel, S., and Vignale, G., *Current reversals in rapidly rotating ultracold Fermi gases*, Phys. Rev. A **89**(6), 063620 (2014). https://doi.org/10.1103/physreva.89.063620