Historical and epistemological perspective of Kaniadakis distribution

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Abstract
The Kaniadakis distribution, proposed by the physicist Georgios Kaniadakis, represents an innovative development within statistical physics, emerging as a generalization of the Maxwell-Boltzmann distribution. This paper proposes a brief historical and epistemological trajectory of the Kaniadakis distribution, addressing its theoretical roots, implications, and the scientific context of its emergence. Initially, we investigate the theoretical foundations that led to the formulation of the Kaniadakis distribution, highlighting how challenges and limitations observed in conventional statistical models motivated the search for new approaches. The Kaniadakis distribution, also known as κ-generalized statistics, emerges as a response to these challenges, introducing a new statistical mechanics consistent with the principles of relativity and thermodynamics. We then explore the impact of this new distribution in various fields, from particle physics to applications in economics, biology, and engineering. We demonstrate how Kaniadakis' approach has provided new perspectives and tools for dealing with systems that exhibit anomalous behaviors, which are not adequately described by traditional statistical theories. Finally, we discuss the epistemological developments of the Kaniadakis distribution within the scientific community. We analyze how its acceptance and integration into different areas of knowledge reflect changes in how scientists understand and model complex phenomena, and the role of theoretical innovation in expanding the boundaries of scientific knowledge.

Keywords: Ensemble, Statistical Physics, Kaniaddakis Distribution.

Resumen
La distribución de Kaniadakis, propuesta por el físico Georgios Kaniadakis, representa un desarrollo innovador dentro de la física estadística, surgiendo como una generalización de la distribución de Maxwell-Boltzmann. Este artículo propone una breve trayectoria histórica y epistemológica de la distribución de Kaniadakis, abordando sus raíces teóricas, implicaciones y el contexto científico de su surgimiento. Inicialmente, investigamos los fundamentos teóricos que llevaron a la formulación de la distribución de Kaniadakis, destacando cómo los desafíos y las limitaciones observados en los modelos estadísticos convencionales motivaron la búsqueda de nuevos enfoques. La distribución de Kaniadakis, también conocida como estadística κ-generalizada, surge como respuesta a estos desafíos, introduciendo una nueva mecánica estadística consistente con los principios de la relatividad y la termodinámica. Posteriormente, exploramos el impacto de esta nueva distribución en diversos campos, desde la física de partículas hasta aplicaciones en economía, biología e ingeniería. Demostramos cómo el enfoque de Kaniadakis ha proporcionado nuevas perspectivas y herramientas para abordar sistemas con comportamientos anómalos, que no se describen adecuadamente en las teorías estadísticas tradicionales. Finalmente, analizamos los desarrollos epistemológicos de la distribución de Kaniadakis dentro de la comunidad científica. Analizamos cómo su aceptación e integración en diferentes áreas del conocimiento reflejan cambios en la forma en que los científicos comprenden y modelan fenómenos complejos, y el papel de la innovación teórica en la expansión de los límites del conocimiento científico.

Palabras clave: Conjunto, Física estadística, Distribución de Kaniadakis.

I. INTRODUCTION

Since its inception by the Austrian physicist Ludwig Boltzmann in 1872, the concept of an ensemble in physics has been a fundamental tool for understanding a wide variety of complex physical phenomena [1]. In his seminal work, Boltzmann introduced the idea of an ensemble as a collection of identical physical systems, each occupying different microstates but all sharing the same total energy. This approach, known as the statistical interpretation of Boltzmann entropy, laid the foundation for ensemble theory and had a transformative impact on the development of statistical physics and statistical mechanics. Ensembles in statistical physics are categorized into several forms, each applicable to different physical contexts [3]. That is, the microcanonical ensemble is used to study systems with well-defined energy, such as crystals at zero temperature, while the canonical ensemble is used for systems in contact with a thermal reservoir of fixed temperature. In addition, the grand canonical ensemble and the Ising ensemble offer specific approaches for systems with different thermodynamic and physical characteristics, respectively. In addition to their solid theoretical foundations, statistical physics ensembles find a wide range of applications in several scientific areas. From Monte Carlo simulations to predict macroscopic properties of materials to the study of complex biological systems such as proteins and DNA, these powerful mathematical structures provide a robust basis for investigating complex phenomena, even when the microscopic details of the systems are unknown.

Statistical physics is a fascinating field that seeks to decipher the secrets of the macroscopic behavior of physical systems from their microscopic properties. Two distinct, yet complementary, approaches have emerged to address this challenging task: extensive and non-extensive statistical physics [4-6]. Extensive statistical physics, also known as classical statistical mechanics, lays the foundations for understanding physical systems such as gases, liquids, and crystalline solids, while non-extensive statistical physics emerges as a response to the challenges presented by systems that defy the laws of classical statistical physics. While extensive statistical physics has its clear limits, non-extensive statistical physics offers a new perspective for understanding complex and unconventional systems.

In the field of statistical physics, statistical distributions have emerged as crucial pillars, enabling scientists to transcend the microscopic complexities of physical systems and thus decipher their macroscopic behaviors. These probabilistic functions, such as the Boltzmann distribution, the Maxwell-Boltzmann distribution, the Fermi-Dirac distribution, and the Bose-Einstein distribution, play a central role in determining the probabilities associated with energy states in diverse systems.

A paradigmatic example lies in the application of the Maxwell-Boltzmann distribution to calculate the average velocity of an ideal gas. This distribution provides crucial insights into the behavior of the gas, such as its diffusion and molecular interactions. In addition, statistical distributions are indispensable tools for calculating thermodynamic properties, studying phase transitions, modeling biological systems, and evaluating material properties.

The importance of statistical distributions in physics is underscored by the fact that they allow physicists to make predictions about the behavior of systems, even when the precise dynamics of individual particles remain unknown. Thus, these tools become essential for unraveling complex phenomena, providing a solid basis for the formulation of theories and predictions. Ultimately, statistical distributions are the foundation upon which rests the ability of physics to explore and understand the intricate details of the physical universe.

On the horizon of contemporary physics, the Kaniadakis distribution [7-11] emerges as a beacon, illuminating the tortuous paths of physical systems that defy the conventions established by classical statistical physics. Introduced in 2001 by the Greek-Italian physicist George Kaniadakis, this statistical distribution generalizes the well-known Boltzmann-Gibbs distribution, offering a multifaceted lens for examining systems with strong fluctuations and long-range interactions.

The versatile applications of the Kaniadakis distribution span a variety of fields, from the physics of critical systems and nonequilibrium systems to the modeling of complex phenomena such as plasmas and gravitational systems. Furthermore, its scope extends beyond the physical domains, finding utility in data analysis in disciplines as diverse as economics, finance, climatology, cosmology, and engineering. Concrete examples include spin systems with long-range interactions, laboratory plasmas, financial markets, and weather patterns. However, it is important to recognize that the Kaniadakis distribution also has its limitations. As an empirical distribution, its form lacks a rigorous theoretical foundation, requiring caution when applying it to specific systems. The inherent complexity of its use can pose significant challenges, demanding a deep understanding and a careful approach.

In summary, the Kaniadakis distribution represents a milestone in contemporary physics, opening doors to the exploration of complex and challenging physical systems. Their insights have the potential to transform our understanding of the physical universe, highlighting the continued need for innovation and exploration in our pursuit of knowledge.

The aim of this paper is to introduce the Kaniadakis distribution, a crucial tool in contemporary statistical physics. We intend to highlight its deep connection through the variation of a deforming parameter, highlighting how it enriches our understanding of complex physical systems. We aim to achieve an accessible conceptual presentation, free from excessive technical formalisms and tedious mathematical demonstrations, thus making the content more accessible and understandable to a broad audience interested in the topic.

II. MAXWELL-BOLTZMANN DISTRIBUTION

A. Brief Context of Kinetic Theory

The kinetic theory of gases aims to describe the macroscopic properties of a gas by means of microscopic quantities that are associated with the particles and molecules that constitute the gas. The beginnings of the atomic theory of matter date back to ancient Greece. After a long period of history and development, today the theory is known as elementary theory and had as pioneers: Daniel Bernoulli (1700-1782), who deduced the Boyle-Robert Boyle law (1627-1691): pV = constant, where p is pressure and V is the volume of the container occupied by the gas and showed that the pressure is proportional to the average of the square of the modulus of the velocities of the molecules; Rudolf Clausius (1822-1888) who introduced the concept of mean free path and James Clerk Maxwell (1831-1879) who introduced the concept of velocity distribution function, known as Maxwell's distribution function and calculated the shear viscosity coefficient of an ideal gas [12].

The foundations of modern kinetic theory were laid by Maxwell, who proposed in 1867 a general transport equation for any macroscopic quantity defined as a function of an average of a microscopic quantity associated with the gas particles. Furthermore, the kinetic theory of gases received a new boost in 1872 when Ludwig Boltzmann (1844-1906) proposed an equation for the distribution function of particle velocities in the form of an integro-differential equation and proved, based on this equation, the second law of thermodynamics [12].

For some applications in the description of a gas of galaxies, electronic plasma, it is assumed that in thermal equilibrium states the distribution function of the constituent particles is described by the classical Maxwell-Boltzmann distribution. It is important to note that the Maxwellian velocity law, as well as the derivation of the famous H-theorem by Boltzmann, were rigorously established only for the model of rigid spheres, neutrons, moving freely and interacting only during collisions. Furthermore, these stimulus results are for all classical Statistical Mechanics, including the theory of Josiah Willard Gibbs (1839-1903) of "Ensembles" [13].

Regarding the development of Thermodynamics, Boltzmann established a connection between the macroscopic (entropy) and microscopic (number of accessible microstates) properties of systems. The connection between statistical mechanics and thermodynamics is established through a microscopic definition for entropy, which is the relevant thermodynamic potential in a system with fixed energy. The fundamental postulate of statistical mechanics establishes that all microscopic states accessible to a closed system in equilibrium are equally probable, since its fundamental theoretical basis is the so-called Boltzmann principle [14], it is represented by the following expression:

$$S = k_B log\Omega \; , \tag{1}$$

where S is the entropy of the system, Ω is the probability obtained with the combinatorial arguments linked to the number of accessible microstates that are compatible with the macrostate and k_B is a universal constant, called by Max Planck, the Boltzmann constant [15]. Classical kinetic theory considers gases to be made up of many particles (of the order of Avogadro's number) that, for the most part, move independently through the volume that contains them. The movement of each particle obeys the laws of mechanics (on the atomic scale, taking quantum effects into account).

The Maxwell-Boltzmann distribution is a good approximation only for ideal gases in thermodynamic equilibrium. Non-ideal gases, in which interactions between molecules are significant, and gases at very low temperatures, where quantum effects become important, may present discrepancies in relation to this distribution.

In this sense, despite these limitations, the Maxwell-Boltzmann distribution is a powerful tool with a wide range of applications in physics, chemistry and engineering. From the calculation of thermodynamic properties to the modeling of planetary atmospheres, its use is fundamental for the advancement of scientific knowledge in several areas. Understanding its limitations is essential for an adequate and accurate application, but this does not diminish the importance of this distribution for modern science and technology.

From this perspective, a statistical distribution is proposed in the case of a gas, in accordance with statistical mechanics for the distribution of molecular velocities. For an ideal gas, some hypotheses are imposed:

I. Isotropic space. That is, all distributions in different directions are equal, all distributions have the same mean velocities, and all directions are equally likely.

II. The x, y, and z components of the velocities are independent, that is,

$$F(V) = f\left(V_x, V_y, V_z\right) = nf(V_x)f\left(V_y\right)f(V_z). \tag{2}$$

Where n represents the particle number density. Also, the term, $f(V_x)dV_x f\left(V_y\right)dV_y f(V_z)dV_z$, represents the probability of finding the particle with the velocity components in the intervals between V_x and $V_x + dV_x$, V_y and $V_y + dV_y$, V_z and V_z+dV_z. It is worth noting that there is no distinction between the directions. Thus, $F(V)$ depends on V_x, V_y and V_z. simply through speed $V = \sqrt{V_x + V_y + V_z}$ and if we take the logarithm, it turns out that:

$$lnF(V) = ln(n) + lnF(V_x) + lnF\left(V_y\right) + lnF(V_z). \tag{3}$$

Successively differentiating equation (3) with respect to V_x, V_y and V_z,, and then the derivative with respect to V_x, results in:

$$\frac{1}{VF(V)}\frac{dF(V)}{dV} = \frac{1}{V_x f(V_x)}\frac{df(V_x)}{dV_x}\,. \tag{4}$$

Making the following definitions, we have that:

$$F(V) \equiv \frac{1}{VF(V)}\frac{dF(V)}{dV}, \tag{5}$$

and

$$f(V_x) \equiv \frac{1}{V_x f(V_x)}\frac{df(V_x)}{dV_x}\,. \tag{6}$$

Therefore, the resulting equation takes the form

$$F(V) = f(V_x)\,, \tag{7}$$

continuing with the differentiation with respect to V_y and V_z implies that:

$$\begin{cases} df\left(V_y\right) = 0 \rightarrow f\left(V_y\right) = c_0, \\ df\left(V_z\right) = 0 \rightarrow f\left(V_z\right) = c_0. \end{cases} \tag{8}$$

Assuming $c_0 = -2a$ in which the signal was introduced to satisfy the normalization condition and the factor 2 for mathematical convenience follows from the equation [12, 13]:

$$\phi(V) = \phi(V_x), \quad (9)$$

$$\phi(V_x) = -2a \rightarrow \frac{1}{V_x f(V_x)}\frac{df(V_x)}{dV_x} = 2aV_x . \quad (10)$$

integrating on both sides, we arrive at the expression $lnf(V_x) = c_1 - 2a(V|x)$, where c_1 is a constant of integration. From this expression and exponentiating on both sides we arrive at the result:

$$f(V_x) = be^{-aV_x{}^x}, \quad (11)$$

where *b* is the integration constant. Since V_x is a positive and bounded function, it follows that the constants *a* and *b* must be positive. The determination of these constants is based on the fact that thermodynamic equilibrium is characterized by two state variables: particle number density *n* and temperature *T*. Thus, to find the V_x component of the velocity in the interval (-∞,+∞):

$$\int_{-\infty}^{+\infty} fV_x dV_x = 1. \quad (12)$$

Solving the integral, we have:

$$b = \sqrt{\frac{a}{\pi}}. \quad (13)$$

Consequently, the determination of the constant a applies to the average energy of $\frac{3}{2}k_B T$ per degree of freedom. Then, it is possible to obtain the following result:

$$a = \frac{M}{2k_B T}. \quad (14)$$

Thus, equation (13) can be rewritten:

$$b = \sqrt{\frac{M}{2\pi k_B T}} . \quad (15)$$

Taking these results into equation (11), the following expression is obtained:

$$f(V_x) = \left(\frac{M}{2\pi k_B T}\right)^{\frac{1}{2}} exp\left(\frac{-MV_x{}^2}{2k_B T}\right), \quad (16)$$

any direction within an infinitesimal volume d^3V of velocity space is given by:

$$\int_0^{2\pi}\int_0^{\pi} f(V)V^2\, sen(\theta)d\theta d\varphi dV = 4\pi f(V)V^2 dV. \quad (17)$$

Where $(V)d^3V$, the probability of a gas molecule having the Maxwell-Boltzmann distribution is given by [14]:

$$f_0(V) \equiv 4\pi(V)V^2 = 4\pi\left(\frac{M}{2\pi k_B T}\right)^{\frac{3}{2}} V^2 exp\left(\frac{-MV^2}{2k_B T}\right). \quad (18)$$

The Maxwell-Boltzmann distribution is a probability distribution that describes the distribution of molecular velocities in an ideal gas in thermodynamic equilibrium [1-2]. It indicates the probability of finding a molecule with a given velocity in a given time interval. Characterized by a bell-shaped curve, the distribution shows that most molecules have velocities close to the average velocity, which is determined by the temperature of the gas and the mass of the molecules.

Thus, from this Maxwell-Boltzmann probability function, equation (above), the average velocity of the particle is determined, such that:

$$\overline{V} = \frac{\int_0^{\infty} Vf_0(V)dV}{\int_0^{\infty} f_0(V)dV} = \sqrt{\frac{8k_B}{\pi M}}. \quad (19)$$

In addition to describing the distribution of molecular velocities, the Maxwell-Boltzmann distribution has several applications in physics and chemistry. That is, it can be used to calculate the viscosity and diffusion of gases, as well as to explain the distribution of light frequencies emitted by atoms in a gas. However, it is important to mention its limitations. From another perspective, the magnitude of the velocity that makes the function $f_0(V)$ a maximum is called the magnitude of the most probable velocity. That is:

$$\left.\frac{df_0(V)}{dV}\right|_{v\,=\,vm_p} = 0. \quad (20)$$

After taking the derivative with respect to V and setting it equal to zero, we obtain:

$$V_{mp} = \sqrt{\frac{2k_B}{M}}. \quad (21)$$

Finally, another characteristic speed is the mean square speed V_{rms}, defined as the square root of the mean value of the speed module, following the following expression:

$$V_{rms} = \sqrt{\overline{V}^2} = \sqrt{\frac{\int_0^{\infty} V^2 f_0(V)dV}{f_0(V)dV}} = \sqrt{\frac{3k_B T}{M}} . \quad (22)$$

From the three speeds, it can be observed that:

$$V_{rms} > \overline{V} > V_{mp} . \quad (23)$$

Figure (1) shows the Maxwell-Boltzmann distribution for a noble gas at different temperatures [13]. Thus, it is possible to observe that, as the temperature increases, the distribution becomes wider and consequently at a more probable speed value.

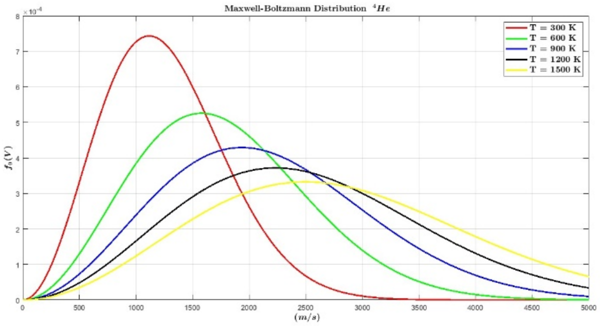

FIGURE 1: Maxwell-Boltzmann distribution for ^{4}He at different temperatures.

Figure (2) illustrates the Maxwell-Boltzmann distribution for some noble gases, considering a temperature of 300 K. It can be seen that the smaller the mass of the gas molecules, the wider the distribution becomes, i.e., causing a shift to the right and the value of the function to decrease. However, once again, it can be seen how much greater the most probable speed becomes.

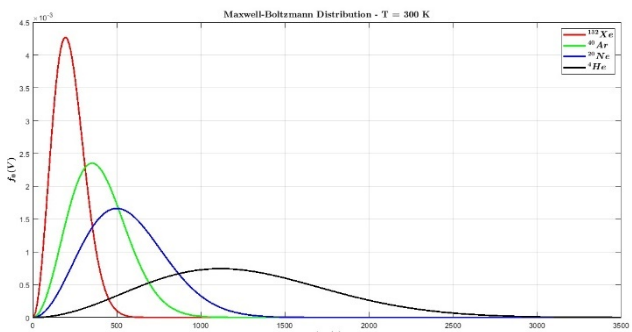

FIGURE 2: Maxwell-Boltzmann distribution for some gases at a temperature of 300 K.

III. KANIADAKIS DISTRIBUTION

Thermodynamics is a phenomenological theory of matter that systematizes the empirical laws on the thermal behavior of macroscopic bodies. From the perspective of statistical mechanics, its objective is to interpret the laws and results of thermodynamics through microscopic information, that is, considering the behavior of a huge number of particles that make up macroscopic bodies [16].

The field of study of statistical mechanics is divided into two fundamental parts: equilibrium and non-equilibrium (or out of equilibrium). The first is well developed and supported in the literature. Through the Gibbs formalism, it is possible to study in detail the macroscopic properties of a system in thermodynamic equilibrium knowing only its Hamiltonian. However, for out-of-equilibrium systems, there is no formulation that describes such systems, even when they are in a steady state. Recently, some proposals have emerged in an attempt to study out-of-equilibrium systems microscopically [17]. Stimulated by systems that exhibit scale invariance, i.e. multifractal systems, the Greek-Brazilian physicist Constantino Tsallis in 1988 presented an article proposing a generalization for Boltzmann-Gibbs statistical mechanics [18]. In this sense, Kaniadakis' proposal also emerges as an alternative for the interpretation of phenomena out of thermal equilibrium. This work is based on the contributions proposed by Giorgio Kaniadakis.

Giorgio, a Greek-Italian engineer and nuclear physicist, presented a new proposal for Maxwell-Boltzmann-Gibbs statistical mechanics in 2001, in which a generalization is expressed through the variation of a deforming parameter. Therefore, it arises naturally within Einstein's special relativity, so that one can see the deformation κ as a purely relativistic effect; κ parameter Kappa defined as a generalization parameter [7, 8, 9, 10]. The generalized Kaniadakis statistics obeys the Kinetical Interaction Principle (KIP), which describes the motion of particles and imposes a form for the entropy of the system. This principle also imposes the form of the generalized entropy associated with the system and allows us to obtain the statistical distribution of these particles (Kaniadakis, 2001b) [8]. In his 2002 article [10], entitled "Statistical mechanics in the context of special relativity", Kaniadakis briefly and consistently presented the foundations of algebra. The deformed κ-statistics can be used to explain a large class of observed phenomena that are described by distribution functions that present power laws. This flexibility makes the Kaniadakis distribution a valuable tool in a variety of scientific and technological fields. From the analysis of astrophysical data to the modeling of traffic systems and the study of earthquakes, the κ distribution has found applications in a wide range of disciplines. Its ability to handle rare events of large magnitude is particularly useful in scenarios where the normal distribution fails to capture the complexity of the data.

Furthermore, the Kaniadakis distribution establishes a connection between several other well-known probability distributions, such as the Boltzmann-Gibbs distribution, the Tsallis distribution, and the Levy distribution, further expanding its scope of application and highlighting its relevance in statistical mechanics and materials science, among other areas.

In the article by Tatsuaki Wada and Antonio Maria Scarfone [23], Kaniadakis distributions are thoroughly explored in their potential to model complex phenomena in statistical physics and natural sciences. By introducing Kaniadakis distributions, based on the hyperbolic arcsine function, the authors offer a valuable generalization of conventional statistical functions, such as the exponential and logarithmic functions, allowing a more accurate description of systems that exhibit power-law tail behavior. Furthermore, by focusing on arsinh-based distributions, they highlight the ability of these distributions to generalize linear constitutive relations, when deformed by κ. These generalizations, as demonstrated by the authors, preserve the original relationships in the limit as κ approaches zero, offering a flexible and robust approach to modeling a variety of physical and natural phenomena.

The numerical simulations presented in the paper provide a powerful empirical confirmation of the effectiveness of Kaniadakis distributions. By analyzing the momentum distribution in a system with kinetic energy deformed by κ, the authors demonstrate that the resulting distribution exhibits characteristics of a k-Gaussian. This numerical analysis not only validates the practical utility of Kaniadakis distributions, but also provides valuable insights into how these distributions can be applied to understand and model complex systems accurately and efficiently. In summary, Wada and Scarfone's study highlights the transformative potential of Kaniadakis distributions in statistical physics and the natural sciences, offering a new perspective for modeling systems with power-law tails and beyond.

Another article that demonstrates the relevance of the distribution is written by Fabio Clementi, which provides a comprehensive and insightful analysis of the Kaniadakis distribution as a valuable tool for examining income and wealth data. Clementi highlights the flexibility of the generalized κ model, introducing its distribution and exploring its basic properties, including its relationship with other statistical distributions widely used in income analysis. By applying the Kaniadakis distribution to real-world data sets on income in Greece, the author carefully compares it with other commonly used distributions, such as the lognormal and Pareto distributions. This comparative analysis reveals the ability of the Kaniadakis distribution to fit real data more accurately in specific cases, highlighting its potential to provide more refined insights into income distribution in real-world settings.

Furthermore, the paper explores broader applications of generalized κ models to income and wealth data in a variety of contexts. Clementi discusses how the Kaniadakis distribution can effectively capture different forms of skewness and heavy tailing, features often present in economic datasets. By providing solid evidence of the successful use of the Kaniadakis distribution in a variety of contexts, the author consolidates the importance of this flexible statistical approach and highlights its significant role in deeper understanding of economic inequality and wealth distribution.

While the Kaniadakis distribution offers many advantages, it also faces significant challenges. Interpreting the κ parameter can be complex, and estimating the distribution parameters can be challenging, especially with small datasets. Furthermore, experimental validation of the Kaniadakis distribution is still ongoing in some areas, highlighting the continued need for research and development and refinement in this area.

In summary, the Kaniadakis distribution represents a promising tool with great potential to advance our understanding and ability to model complex systems in several areas of science and technology. Its flexibility, accuracy, and generality make it a significant contribution to the arsenal of tools available to scientists and engineers, offering new perspectives and insights in a variety of fields.

A. Mathematical definitions

The Kaniadakis statistic was introduced by Giorgio Kaniadakis in 2001 in the paper Kaniadakis (2001b) [8]. This statistical framework incorporates a new one-parameter warped exponential function, κ. The behavior of this statistic is governed by the following characteristics [7, 8, 9, 10]:

$$exp_{\kappa}(x)exp_{\kappa}(-x) = 1, \tag{24}$$

where the deformed exponential function obeys the following limit:

$$\lim_{\kappa\to 0} exp_{\kappa}(x) \to exp(x). \tag{25}$$

Considering these conditions and the statistical evolution, the deformed exponential function assumes the following expression:

$$f(x) = \left(\sqrt{1+\kappa^2x^2}+\kappa x\right)^{\frac{1}{\kappa}}, \text{ if } 0<\kappa<1 \text{ and } f(x) = exp(x) \text{ if } \kappa \to 0. \tag{26}$$

The function in relation to the change in the value of the κ factor can be observed in Figure (3):

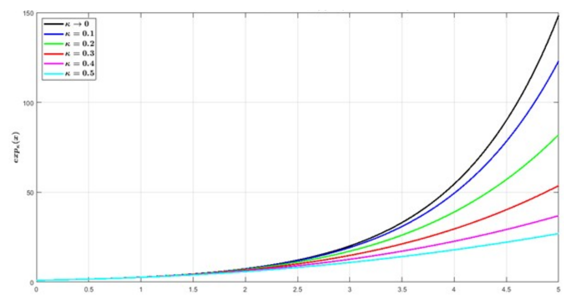

FIGURE 3: Behavior of the expression $exp_{\kappa}(x)$ for different values of κ.

A similar way to represent the Kaniadakis distribution [7-10], as per the following expression:

$$exp_{\kappa}(x) = exp\left[\frac{1}{\kappa}arcsin(\kappa x)\right]. \tag{27}$$

B. Applications of Kaniadakis Distribution

Despite its recent emergence, this statistical model has gained significant acceptance in several areas within the field of physics and beyond. Here, we present a selection of studies that have used this statistical framework to clarify their respective systems.

In Luciano [20], the authors state that gravitational and cosmological scenarios based on the Kaniadakis entropy (deformed by κ) have been considered, resulting in generalized models that predict a richer phenomenology compared to their standard Maxwell-Boltzmann counterparts. The work in the paper includes a review of the implications of the entropy of κ in Holographic Dark Energy, Entropic Gravity, black hole thermodynamics, Loop Quantum Gravity, and Big Bang Nucleosynthesis.

In Kaniadakis et al. [9], the authors state that a growing number of applications in different fields of research are beginning to prove the relevance and effectiveness of k-statistics in fitting empirical data. In the paper, the authors use k-statistics to formulate a statistical approach for epidemiological analysis. They validate the theoretical results by fitting the derived Weibull distributions with data from the 1417 plague pandemic in Florence, as well as data from the COVID-19 pandemic in China over the entire cycle. In Kaniadakis [7], the authors present a proof of the quantum H theorem using Kaniadakis entropy and present a quantum version of the second law of thermodynamics consistent with Kaniadakis statistics.

In Clementi et al. [21], we observe an approach using κ-statistics for the study of economics through the analysis of income distribution and inequality.

In Martinez and de Abreu [22], the authors highlight the contribution of Kaniadakis entropy to nuclear reactor physics through the Doppler broadening function.

These articles cover diverse fields, however, there is a proliferation of research using this statistic in all the areas presented, such as [7, 8, 9, 10, 22-26].

However, to our knowledge, no one has yet discussed the circumstances under which the Kaniadakis distribution should be preferred over the Tsallis distribution, or vice versa [27, 28].

The Kaniadakis distribution [7], is given by:

$$f_\kappa(V) = A_\kappa exp_\kappa\left(\frac{-MV^2}{2k_BT}\right). \quad (28)$$

Where, A_κ is a parameter which dependents on κ. Therefore, the probability of finding a particle with velocity V, in any direction within the volume element d^3V of the velocity space, is given by:

$$\int_0^{2\pi}\int_0^{\pi} f_\kappa(V)V^2\, sen(\theta)d\theta d\varphi dV = 4\pi f_\kappa(V)V^2 dV. \quad (29)$$

Therefore the Kaniadakis distribution is given by:

$$f_{0_\kappa}(V) \equiv 4\pi f_\kappa(V) = 4\pi A_\kappa V^2 exp_\kappa\left(\frac{-MV^2}{2k_BT}\right), \quad (30)$$

Thus:

$$f_{0_\kappa}(V) = 4\pi A_\kappa V^2 exp_\kappa\left(\frac{-MV^2}{2k_BT}\right). \quad (31)$$

Note that from the normalization condition, the parameter A_κ can be determined, such that:

$$\int_{-\infty}^{+\infty} f_{0_\kappa}(V)dV = 4\pi A_\kappa \int_0^{+\infty} V^2\, exp_\kappa\left(\frac{-MV^2}{2k_BT}\right)dV = 1. \quad (32)$$

Performing the following change of variable, we have:

$$u = \left(\frac{MV^2}{2k_BT}\right) \to du = \left(\frac{MV}{k_BT}\right)dV. \quad (33)$$

After simple substitution, you get the following expression:

$$2\pi A_\kappa \left(\frac{2k_BT}{M}\right)^{\frac{3}{2}} \int_0^{+\infty} (u)^{\frac{1}{2}}\, exp_\kappa(u)du = 1. \quad (34)$$

Furthermore, using the Mellin transform [29] and comparing it with the integral (34), it is concluded that:

$$A_\kappa \equiv \left(\frac{|\kappa|M}{\pi k_BT}\right)^{\frac{3}{2}}\left(1+\frac{3}{2}|\kappa|\right)\frac{\Gamma\left(\frac{1}{2|\kappa|}+\frac{3}{4}\right)}{\Gamma\left(\frac{1}{2|\kappa|}-\frac{3}{4}\right)}; \quad \text{for} \quad |\kappa| < \frac{2}{3}. \quad (35)$$

Being A_κ the normalization parameter of $f_{0_\kappa}(V)$, from the Kaniadakis distribution function it is possible to find the fastest speed, $\left(V_{\kappa_{mp}}\right)$ [30]. That is:

$$\frac{df_{0_\kappa}(V)}{dV} = 4\pi A_\kappa [2V - \frac{MV^3}{k_BT\sqrt{\kappa^2\left(\frac{MV^2}{\pi k_BT}\right)^2+1}}] exp_\kappa\left(\frac{-MV^2}{2k_BT}\right) = 0. \quad (36)$$

Solving it we find that:

$$V_{\kappa_{mp}} = \left(\frac{2k_BT}{M\sqrt{1-\kappa^2}}\right)^{\frac{1}{2}}. \quad (37)$$

Applying the same reasoning to the case of average speed, as proposed by [30], the following result follows:

$$\overline{V_\kappa} = \int_0^{+\infty} V f_{0_\kappa}(V)dV = 4\pi A_\kappa \int_0^{+\infty} V^3\, exp_\kappa\left(\frac{-MV^2}{2k_BT}\right)dV. \quad (38)$$

Substituting into equation (33):

$$\overline{V_\kappa} = 8\pi A_\kappa \left(\frac{k_BT}{M}\right)^2 \int_0^{+\infty} u\, exp_\kappa(-u)du. \quad (39)$$

With this, the equation is replaced again in the Mellin transform [13], which can be represented as follows:

$$\overline{V_\kappa} = \frac{8\pi A_\kappa}{1-4\kappa^2}\left(\frac{k_BT}{M}\right)^2; \text{ for } |\kappa| < \frac{2}{3}. \quad (40)$$

After substituting equation (35), the following relationship is obtained:

$$\overline{V_\kappa} = \left(\frac{2\sqrt{2}}{1-4\kappa^2}|\kappa|^{\frac{3}{2}}\right)\left(1+\frac{3}{2}|\kappa|\right)\frac{\Gamma\left(\frac{1}{2|\kappa|}+\frac{3}{4}\right)}{\Gamma\left(\frac{1}{2|\kappa|}-\frac{3}{4}\right)}\left(\frac{8k_BT}{\pi M}\right)^{\frac{1}{2}}; \text{ for } |\kappa| < \frac{1}{2} \quad (41)$$

Taking these aspects into consideration, therefore, the mean quadratic speed is given by:

$$\overline{V_\kappa}^2 = \int_0^{+\infty} V^2 f_{0_\kappa}(V)dV = 4\pi A_\kappa \int_0^{+\infty} V^4\, exp_\kappa\left(\frac{-MV^2}{2k_BT}\right)dV \quad (42)$$

Since, using the equations defined in (33), (35) and the Mellin transform [13, 31, 32], we have that:

$$\overline{V_\kappa}^2 = 2 \ (\text{m/s})^2 \text{ for } |\kappa| < \frac{2}{5}. \quad (43)$$

Finally, for the mean square velocity, $V_{\kappa_{rms}}$, defined as the root mean square of the velocity, that is:

$$\bar{V}^2{}_{\kappa_{rms}} = \left[\frac{2(4|\kappa|^3(2+3|\kappa|)}{(4-|\kappa|)^2(4-25|\kappa|^2)(2-3|\kappa|)}\right]^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2|\kappa|}+\frac{3}{4}\right)}{\Gamma\left(\frac{1}{2|\kappa|}-\frac{3}{4}\right)} \left(\frac{3k_BT}{M}\right)^{\frac{1}{2}}. \tag{44}$$

Therefore, it is also concluded that:

$$V_{\kappa_{rms}} < \overline{V_\kappa} < V_{\kappa_{mp}}. \tag{45}$$

It can be observed that the results obtained by equations (37), (41) and (44) were reduced to the results given by the Maxwell-Boltzmann distribution, equations (19), (21) and (22), when κ→0. The same occurs with the Kaniadakis distribution itself, which tends towards the Maxwell-Boltzmann distribution, when κ→0.

To evaluate the behavior of the Kaniadakis distribution, following equation (31) and with different temperatures and κ-deformations, as shown in Figure (4) for the ^{4}He isotope.

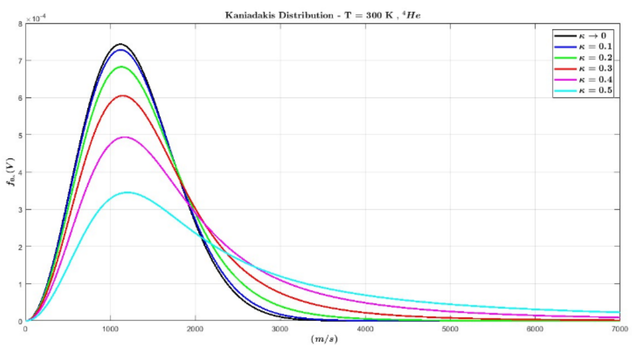

FIGURE 4: Velocity distribution of Kaniadakis ^{4}He at temperature of 300 K and κ-strains.

In Figure (4), the behavior of the probability function of velocities is observed for a set governed by the Kaniadakis statistical distribution, considering 4He nuclides and different temperatures. In fact, it was observed that the Kaniadakis distribution resumes the Maxwell-Boltzmann distribution when κ → 0.

VI. CONCLUSIONS

In view of the above, this study explored the main contributions of the Kaniadakis distribution to statistical mechanics and its wide applications in several fields of knowledge. By revisiting the historical and epistemological trajectory of this distribution, we highlight how the theoretical foundations and motivations for its formulation arose from the limitations observed in traditional statistical models. The Kaniadakis distribution, as a generalization of conventional statistics, brought new perspectives and tools for understanding complex systems that present anomalous behaviors. The Kaniadakis distribution is particularly useful in modeling complex phenomena, such as rare events of large magnitude, where normal distributions fail or do not present very satisfactory results. This distribution also connects several other known probability distributions, expanding its scope of application and relevance in statistical mechanics and other areas of knowledge. Although recent, the statistical model based on Kaniadakis entropy has been widely accepted and applied in several fields of physics, engineering and data analysis. Its flexibility and ability to provide a richer description of phenomena compared to conventional Maxwell-Boltzmann models highlight its potential.

In summary, Kaniadakis statistics offers a powerful theoretical and practical tool for modeling complex systems, providing new insights and possibilities in various fields of human knowledge. The continued expansion of applications and the need for comparative studies highlight the relevance and transformative potential of this statistical approach. Furthermore, including the Kaniadakis distribution in the physics curriculum can provide students with a more comprehensive view of generalized statistics and its applications in diverse fields, from particle physics to economics and biology. Furthermore, by discussing the challenges faced and the solutions proposed by Kaniadakis, educators can encourage critical and creative thinking among students, fostering an appreciation for the dynamic and evolving nature of science.

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