Analytical solutions to the fractional wave equation with variable dielectric function

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Abstract

The fractional wave equation is presented as a generalization of the wave equation when arbitrary fractional order derivatives are involved. We have considered variable dielectric environments for the wave propagation phenomena. The Jumarie's modified Riemann-Liouville derivative has been introduced and the solutions of the fractional Riccati differential equation have been applied to construct analytical solutions of the fractional wave equation. New family of exact solutions has been found for the fractional wave equation. These new solutions are compared with that obtained previously in the literature for the case of integer order derivatives. The results show how powerful can result the fractional calculus when is applied to many different physical situations.

Keywords: Fractional wave equation, Variable dielectric environments for wave propagation, Analytical solution for the fractional wave equation.

Resumen

Se presenta la ecuación de onda de orden fraccionario como una generalización de la ecuación de onda cuando se tienen derivadas de orden fraccionario arbitrario. Se consideran medios dieléctricos variables para la propagación de ondas. Se emplea la derivada fraccionaria de Riemann-Liouville modificada por Jumarie y se aplican las soluciones de la ecuación diferencial fraccionaria de Riccati para obtener soluciones analíticas para la ecuación de onda fraccionaria. Una familia nueva de soluciones para la ecuación de onda fraccionaria se ha obtenido. Estas nuevas soluciones se comparan con las soluciones obtenidas previamente en la literatura para el caso de derivadas de orden entero. Los resultados muestran lo poderoso que resulta el cálculo fraccionario cuando se aplica a diversas situaciones físicas.

Palabras clave: Ecuación de onda fraccionaria, Medios dieléctricos variables para la propagación de ondas, Soluciones analíticas para la ecuación de onda fraccionaria.

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I. INTRODUCTION

Fractional differential equations are generalizations of classical differential equations of integer order. In recent years, nonlinear fractional differential equations have gained considerable interest. It is caused by the development of the theory of fractional calculus itself but also by the applications of such constructions in various sciences such as physics, engineering, biology and others areas [1, 2, 3, 4, 5, 6, 7]. Among the investigations for fractional differential equations, research for seeking exact solutions is an important topic as well as applying them to practical problems [8, 9, 10, 11, 12, 13].

The exact solutions of the electromagnetic wave equation for inhomogeneous medium for physically relevant dielectric function have attracted much attention of the physicist since many years ago. Considerable effort [14] has been made in order to obtain exact solutions to the electromagnetic wave equation for inhomogeneous medium.

In this paper, some basic properties of the fractional calculus have been successfully employed to obtain the analytical solution of the fractional wave-like equation where we have incorporated variable dielectric environments for wave propagation into inhomogeneous medium. Here we have considered a special dielectric function of fractional polynomial form:

$$\varepsilon(\omega,z)=\varepsilon(z)=\left(\frac{a}{b+z^{\alpha}}\right)^{2}. \qquad (1)$$

The method for solving the fractional wave equation is based on the modified Riemann-Liouville fractional derivative of order α [15, 16] and the analytical solutions for the fractional Riccati differential equation without using any restrictive assumption [17]. The exact solutions for the electric fields of the fractional wave equation are expressed in terms of simple polynomial functions in the fractional variable z, associated with the wave propagation direction.

From historical point of view fractional calculus may be described as an extension of the concept of a derivative operator from integer order n to arbitrary order α, where α is a real or complex number:

$$\frac{d^n}{dx^n} \to \frac{d^\alpha}{dx^\alpha} \quad . \tag{2}$$

The physicists have been attracted to the fractional differential equations that have been applied in several areas, like: wave and diffusion equations, Schrödinger equation, Yang-Mills theory, nuclear and particle physics [18].

The recent appearance of fractional differential equations and their applications in physical-mathematical problems make necessary to investigate the methods for the solution for such equations (analytical and numerical) and we hope that this work is a step in this direction. We present, for the interested students and the professor research as well, a concise example of a fractional wave equation in the electromagnetic wave propagation with analytical solution.

In order to give a pedagogical approach to this problem, in section II we first present a brief introduction (with out a rigorous proof) to the fractional derivatives and the elemental properties of the fractional derivatives. In the literature there are several definitions for the fractional derivatives, here we will only consider the Jumarie's modified Riemann-Liouville definition for fractional derivative [15, 16], because the fractional derivative defined in this way results to be very useful when analytical solutions for fractional differential equations are investigated. After this we present the general analytical solutions for the Riccati fractional differential equation [17]. Then in section III we introduce the electromagnetic wave propagation into an inhomogeneous medium where some analytical solutions to this problem have been obtained previously [14]. Next in section III.B we present the application of the fractional calculus to solve the inhomogeneous fractional order wave equation when the dielectric function takes the fractional position dependence of the equation (1). In section IV, we discuss the reliability of the proposed method and the exact solutions are compared with the results reported in the literature [14], when only the inhomogeneous integer order wave propagation has been considered. Finally in section V some conclusions are presented.

II. FRACTIONAL CALCULUS (BASIC IDEAS)

If we consider an application of differential or integral calculus simply as mapping from a given function set f onto another set g, e.g.,

$$g(x) = \frac{d}{dx} f(x) \; . \tag{3}$$

Then in general from this relation we cannot deduce any valid information on a possible similarity of a function and its derivative.

Therefore it is surprising and remarkable that for particular function classes we observe a very simple relationship in respect of their derivatives.

It is easy to show that for the exponential, the trigonometric and the powers functions a simple rule can be written for all $n \in N$ [18]:

$$\begin{aligned}
&\frac{d^n}{dx^n} e^{kx} = k^n e^{kx}, \\
&\frac{d^n}{dx^n} \sin(kx) = k^n \sin\left(kx + \frac{n\pi}{2}\right), \\
&\frac{d^n}{dx^n} x^k = \frac{k!}{(k-n)!} x^{k-n} \quad .
\end{aligned} \tag{4}$$

For arbitrary order n apparently a kind of self similarity emerges, e.g., all derivatives of the exponential lead to exponentials, all derivatives of trigonometric functions lead to trigonometric functions. Since the derivative is given in a closed form it is straightforward to extend this rule from integer derivative coefficients $n \in N$ to real and even imaginary coefficients α and postulate the fractional derivative as:

$$\begin{aligned}
&\frac{d^\alpha}{dx^\alpha} e^{kx} = k^\alpha e^{kx}, && k \ge 0 \\
&\frac{d^\alpha}{dx^\alpha} \sin(kx) = k^\alpha \sin\left(kx + \frac{\alpha\pi}{2}\right), && k \ge 0 \\
&\frac{d^\alpha}{dx^\alpha} x^k = \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)} x^{k-\alpha}, && x \ge 0,\ k \ne -1,-2,-3,...
\end{aligned} \tag{5}$$

We restrict to $k \ge 0$ and $x \ge 0$ respectively to ensure the uniqueness of the fractional derivative definition [8-13].

Now we consider the formal definition for the Jumarie's modified Riemann-Liouville fractional derivative of order α ($D^\alpha{}_x$) [15, 16]:

$$D_x^\alpha f(x) = \begin{cases}
\frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{\left[f(\xi) - f(0)\right]}{(x-\xi)^{\alpha+1}} d\xi \,, \\
\qquad \text{for } \alpha < 0 \\
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{\left[f(\xi) - f(0)\right]}{(x-\xi)^{\alpha}} d\xi, \\
\qquad \text{for } 0 < \alpha < 1 \\
\left[f^{(\alpha-n)}(x)\right]^{(n)}, \\
\qquad \text{for } n \le \alpha \le n+1,\ n \ge 1.
\end{cases} \tag{6}$$

Some properties for the proposed modified Riemann–Liouville derivative are [15, 16]:

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha},$$
$$D_x^\alpha c = 0, \quad \alpha \geq 0 \quad c = const.$$
$$D_x^\alpha\left(cf(x)\right) = c\left(D_x^\alpha f(x)\right), \quad \alpha \geq 0 \quad c = const. \tag{7}$$
$$D_x^\alpha\left(f(x)g(x)\right) = g(x)\left(D_x^\alpha f(x)\right) + f(x)\left(D_x^\alpha g(x)\right),$$
$$D_x^\alpha f\left[g(x)\right] = f_g'\left[g(x)\right] D_x^\alpha g(x) \; .$$

Other important result for the research of analytical solutions of the fractional wave equation is the exact solution of the fractional Riccati equation:

$$D_\xi^\alpha \phi = \sigma + \phi^2 \quad , \tag{8}$$

where σ is a constant. By using the generalized exp-function method via Mittag-Leffler function, Zhang et al. [19], obtained the following solutions of the fractional Riccati equation (8):

$$\phi(\xi) = \begin{cases} -\sqrt{-\sigma}\tanh_\alpha\left(\sqrt{-\sigma}\xi\right), & \sigma < 0, \\ -\sqrt{-\sigma}\coth_\alpha\left(\sqrt{-\sigma}\xi\right), & \sigma < 0, \\ \sqrt{\sigma}\tan_\alpha\left(\sqrt{\sigma}\xi\right), & \sigma > 0, \\ \sqrt{\sigma}\cot_\alpha\left(\sqrt{\sigma}\xi\right), & \sigma > 0, \\ -\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}, \quad \omega = const. & \sigma = 0 \; , \end{cases} \tag{9}$$

where the generalized hyperbolic and trigonometric functions are defined as:

$$\begin{aligned} \sinh_\alpha(x) &= \frac{E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)}{2}, \\ \cosh_\alpha(x) &= \frac{E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)}{2}, \\ \tanh_\alpha(x) &= \frac{\sinh_\alpha(x)}{\cosh_\alpha(x)}, \\ \coth_\alpha(x) &= \frac{\cosh_\alpha(x)}{\sinh_\alpha(x)}, \\ \sin_\alpha(x) &= \frac{E_\alpha(ix^\alpha) - E_\alpha(-ix^\alpha)}{2}, \\ \cos_\alpha(x) &= \frac{E_\alpha(ix^\alpha) + E_\alpha(-ix^\alpha)}{2}, \\ \tan_\alpha(x) &= \frac{\sin_\alpha(x)}{\cos_\alpha(x)}, \\ \cot_\alpha(x) &= \frac{\cos_\alpha(x)}{\sin_\alpha(x)}, \end{aligned} \tag{10}$$

where $E_\alpha(z)$ is the Mittag-Leffler function, given as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}. \tag{11}$$

III. FRACTIONAL WAVE EQUATION IN INHOMOGENEOUS MEDIUM

In several cases the propagation or transmission of a physical quantity can be modeled by a wave equation in which the velocity is a function of the propagation coordinate: for instance, the cases of electromagnetic waves in normal incidence on a region whose electric permeability depends on the position in the medium, thin film coating of optical surfaces where reflection is of practical interest, radio wave reflection, propagation or transmission of electromagnetic field in the ionosphere, optical systems with variable index of refraction, etc. For this kind of system, several dielectric profiles have been solved analytically: the inverse squared profile, exponential, linear and quadratic polynomials (*e.g.*, V. Ginzburg [14] and references therein).

The propagation of waves in inhomogeneous isotropic media involves a very wide range of possibilities, which arise mainly from the specific form of the dielectric function ε. It is necessary to state the problem more definitely, here we consider a medium which consist of plane-parallel layers. The propagation of waves in a plane-parallel layer medium may conveniently be first considered for the particular case of a wave incident normally on a layer of an inhomogeneous medium. For this case we may consider electric fields of the form:

$$E(z,t) = E(z)e^{i\omega t}, \tag{12}$$

where ω is the angular frequency and z is the wave propagation direction. $E(z)$ is the electric field perpendicular to the wave propagation direction ($E_x(z)$ or $E_y(z)$), the component $E_z(z)$ is taken to be zero. The electric field satisfies the wave equation:

$$\begin{aligned} &\frac{\partial^2 E(z,t)}{\partial z^2} = \frac{\varepsilon(\omega,z)}{c^2}\frac{\partial^2 E(z,t)}{\partial t^2} \\ &\Rightarrow \\ &\frac{d^2 E(z)}{dz^2} + \frac{\omega^2 \varepsilon(\omega,z)}{c^2} E(z) = 0, \end{aligned} \tag{13}$$

$\varepsilon(\omega,z)$ is the dielectric function inside the inhomogeneous medium. This wave equation for arbitrary $\varepsilon(\omega,z)$ has no solution which can be written in terms of known functions, the particular cases where this can be done are of considerable interest [14]. For example, in the case of a linear form $\varepsilon(\omega,z)= \varepsilon(z)=a+bz$, the solution of the equation (13) can be expressed in terms of known functions of order 1/3 or Airy functions. For a parabolic form $\varepsilon(z)=a+bz^2$, the solution can be expressed in terms of parabolic cylinder functions (Weber functions) [14]. The solutions for $\varepsilon(z)=$

$(a+bz)^m$ with integral m can be expressed in terms of Bessel functions; for m = -2 the solution is a power function.

A. Wave propagation in a inhomogeneous medium with a dielectric function $\varepsilon(z)=(a/(b+z))^2$

We shall now discuss one of the simplest exact solutions for the equation (13), namely the special case where:

$$\varepsilon(\omega,z)=\varepsilon(z)=\left(\frac{a}{b+z}\right)^2 \quad , \tag{14}$$

which is of interest because the exact solution is expressible in terms of elementary functions. For the wave equation:

$$\frac{d^2E}{dz^2}+\frac{\omega^2}{c^2}\left(\frac{a}{b+z}\right)^2 E=0 \quad , \tag{15}$$

it is easily verified, by direct substitution in the above equation, that the solution is

$$\begin{gathered} E(z)=C_1(b+z)^{r_1}+C_2(b+z)^{r_2} \quad , \\ r_{1,2}=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\left(\frac{\omega a}{c}\right)^2}=\frac{1}{2}\pm i\sqrt{\left(\frac{\omega a}{c}\right)^2-\frac{1}{4}} \quad . \end{gathered} \tag{16}$$

As an example, let us consider reflection from a layer of the type of the equation (14):

$$\begin{gathered} \varepsilon=1 \quad \text{for } z\le\Lambda \quad \text{(medium 1)}, \\ \varepsilon=\left(\frac{\Lambda}{z}\right)^2 \quad \text{for } z>\Lambda \quad \text{(medium 2)}, \end{gathered} \tag{17}$$

with b = 0 and $a = \Lambda$ in the equation (14) , see Figure 1.

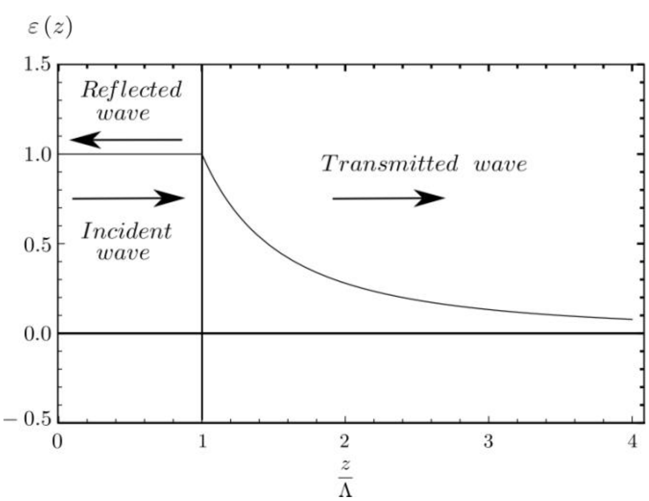

FIGURE 1. The dielectric function ε(*z*) given by the equation (17) with Λ=12.5.

Let the wave be incident from medium 1, a vacuum, where the field has the form

$$E_1=e^{-i\frac{\omega}{c}(z-\Lambda)}-R\left(e^{i\frac{\omega}{c}(z-\Lambda)}\right) \quad , \tag{18}$$

here we consider both the incident wave traveling to the right and the reflected wave (R) traveling backwards to the left. In medium 2 the field is

$$\begin{aligned} E_2 &= Az^{1/2-i\sqrt{(\omega\Lambda/c)^2-1/4}}=Az^{1/2}z^{-i\sqrt{(\omega\Lambda/c)^2-1/4}} \\ &\equiv Az^{1/2}z^{-i\beta}=Az^{1/2}e^{-i\beta\ln z} \quad , \end{aligned} \tag{19}$$

since in the equation (16) we must put b=0, $a = \Lambda$; we have also used the fact that in medium 2 there is only a wave travelling away from the boundary. It is assumed that the wave can be propagated, which implies that $\omega\Lambda/c > 1/2$. At the boundary we must have $E_1=E_2$ and $(dE_1/dz)=(dE_2/dz)$, whence:

$$\begin{gathered} R=1-A(\Lambda)^{1/2-i\beta}=i\frac{1}{2\left(\frac{\omega\Lambda}{c}+\beta\right)}, \\ |R|=\frac{1}{2\left(\frac{\omega\Lambda}{c}+\beta\right)}, \\ A=\frac{2(\Lambda)^{-1/2+i\beta}}{1+i\frac{c}{\omega\Lambda}\left(\frac{1}{2}-i\beta\right)}, \\ \beta=\sqrt{\left(\frac{\omega\Lambda}{c}\right)^2-\frac{1}{4}} \quad . \end{gathered} \tag{20}$$

B. Fractional wave propagation in inhomogeneous medium with a dielectric function $\varepsilon(z)=(a/(b+z^\alpha))^2$

Now if we consider the generalization of the equation (13) for an arbitrary fractional order we obtain:

$$\begin{gathered} D_z^{2\alpha}E(z,t)=\frac{\varepsilon(\omega,z)}{c^2}\frac{\partial^2E(z,t)}{\partial t^2} \\ \Rightarrow \\ D_z^{2\alpha}E(z)+\frac{\omega^2\varepsilon(\omega,z)}{c^2}E(z)=0, \end{gathered} \tag{21}$$

in the special case where:

$$\varepsilon(\omega,z)=\varepsilon(z)=\left(\frac{a}{b+z^\alpha}\right)^2 \quad . \tag{22}$$

For the wave equation:

$$D_z^{2\alpha} E + \frac{\omega^2}{c^2}\left(\frac{a}{b+z^\alpha}\right)^2 E = 0, \qquad (23)$$

it is easy to show that the solution for this equation is given by:

$$E(z) = A\left(b+z^\alpha\right)^s \quad . \qquad (24)$$

We can verified the above result by taking into account the properties of the Jumarie's modified Riemann-Liouville fractional derivative (see equation (7))

$$\begin{aligned} D_z^\alpha\left(\left(b+z^\alpha\right)^s\right) &= D_z^\alpha\left(\left(\frac{1}{b+z^\alpha}\right)^{-s}\right) = \\ &= -s\left(\frac{1}{b+z^\alpha}\right)^{-s-1} D_z^\alpha\left(\left(\frac{1}{b+z^\alpha}\right)\right) \quad , \end{aligned} \qquad (25)$$

and noting that

$$\begin{aligned} D_z^\alpha\left(\left(\frac{1}{b+z^\alpha}\right)\right) &= -\frac{1}{\Gamma(1+\alpha)} D_z^\alpha\left(-\frac{\Gamma(1+\alpha)}{b+z^\alpha}\right) , \\ &= -\frac{1}{\Gamma(1+\alpha)}\left(\frac{\Gamma(1+\alpha)}{b+z^\alpha}\right)^2 , \\ &= -\Gamma(1+\alpha)\left(\frac{1}{b+z^\alpha}\right)^2 , \end{aligned} \qquad (26)$$

where we have taken into account that the function $\phi(z)$= $-\Gamma(1+\alpha)/(b+z^\alpha)$ is one of the analytical solutions (9) of the fractional Ricatti equation (8), therefore we obtain:

$$\begin{aligned} D_z^\alpha\left(\left(b+z^\alpha\right)^s\right) &= D_z^\alpha\left(\left(\frac{1}{b+z^\alpha}\right)^{-s}\right), \\ &= s\left(\frac{1}{b+z^\alpha}\right)^{-s-1}\Gamma(1+\alpha)\left(\frac{1}{b+z^\alpha}\right)^2 , \\ &= s\Gamma(1+\alpha)\left(\frac{1}{b+z^\alpha}\right)^{-s+1} , \end{aligned} \qquad (27)$$

and

$$\begin{aligned} D_z^{2\alpha}\left(\left(\frac{1}{b+z^\alpha}\right)^{-s}\right) &= s\Gamma(1+\alpha) D_z^\alpha\left(\left(\frac{1}{b+z^\alpha}\right)^{-s+1}\right) \\ & s(-s+1)\Gamma(1+\alpha)\left(\frac{1}{b+z^\alpha}\right)^{-s} D_z^\alpha\left(\frac{1}{b+z^\alpha}\right), \\ &= s(s-1)\left(\Gamma(1+\alpha)\right)^2\left(\frac{1}{b+z^\alpha}\right)^{-s}\left(\frac{1}{b+z^\alpha}\right)^2 \quad , \end{aligned} \qquad (28)$$

and from equation (23) we found:

$$\begin{aligned} & D_z^{2\alpha} E + \frac{\omega^2}{c^2}\left(\frac{a}{b+z^\alpha}\right)^2 E \\ &= s(s-1)\left(\Gamma(1+\alpha)\right)^2\left(\frac{1}{b+z^\alpha}\right)^{-s+2} \\ &+ \frac{\omega^2}{c^2}\left(\frac{a}{b+z^\alpha}\right)^2\left(\frac{1}{b+z^\alpha}\right)^{-s} = 0 \quad , \end{aligned} \qquad (29)$$

and the solution to the wave equation (23) is obtained by solving the quadratic equation:

$$s(s-1)\left(\Gamma(1+\alpha)\right)^2 + \frac{\omega^2 a^2}{c^2} = 0 \quad , \qquad (30)$$

the solutions of the above equation are given by

$$\begin{aligned} s_{1,2} &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - \left(\frac{\omega a}{\Gamma(1+\alpha)c}\right)^2} \\ &= \frac{1}{2} \pm i\sqrt{\left(\frac{\omega a}{\Gamma(1+\alpha)c}\right)^2 - \frac{1}{4}} \quad , \end{aligned} \qquad (31)$$

and finally the solution for the fractional wave equation (23) is given by

$$E(z) = A\left(b+z^\alpha\right)^{s_1} + B\left(b+z^\alpha\right)^{s_2} \quad . \qquad (32)$$

Let us now consider the reflection from a layer of the type (14):

$$\begin{aligned} &\varepsilon = 1 \quad \text{for } z \le \Lambda \quad \text{(medium 1)}, \\ &\varepsilon = \left(\frac{\Lambda}{z}\right)^{2\alpha} \quad \text{for } z > \Lambda \quad \text{(medium 2)}, \end{aligned} \qquad (33)$$

with b=0 (see figure 1). Let the wave be incident from medium 1, a vacuum $z \le \Lambda$, where the field has the form

$$\begin{aligned} E_1(z) &= E_\alpha\left(-i\frac{\omega}{c}\left(z^\alpha - \Lambda^\alpha\right)\right) \\ &- R\left[E_\alpha\left(i\frac{\omega}{c}\left(z^\alpha - \Lambda^\alpha\right)\right)\right] , \end{aligned} \qquad (34)$$

and E_α is the Mittag-Leffler function defined previously (see equation (11)). The equation (34) is the fractional order generalization for the incident wave of the equation (18). The electric field given in terms of the Mittag-Leffler function satisfies the following fractional wave equation:

$$\begin{aligned} & D_z^{2\alpha} E_\alpha\left(-i\frac{\omega}{c}\left(z^\alpha - \Lambda^\alpha\right)\right) \\ &+ \frac{\omega^2}{c^2} E_\alpha\left(-i\frac{\omega}{c}\left(z^\alpha - \Lambda^\alpha\right)\right) = 0 \quad . \end{aligned} \qquad (35)$$

In medium 2 the field is

$$\begin{aligned} E_2(z) &= A\left(z^{\alpha}\right)^{1/2 - i\sqrt{\left(\omega\Lambda^{\alpha}/c\Gamma(1+\alpha)\right)^2 - 1/4}}, \\ &= Az^{(1/2)\alpha} z^{-i\alpha\sqrt{\left(\omega\Lambda^{\alpha}/c\Gamma(1+\alpha)\right)^2 - 1/4}}, \\ &\equiv Az^{(1/2)\alpha} z^{-i\beta\alpha} = Az^{(1/2)\alpha} e^{-i\beta\alpha \ln z}, \end{aligned} \tag{36}$$

since in the equation (32) we must put b=0, $a = \Lambda^{\alpha}$; we have also used the fact that in medium 2 there is only a wave travelling away from the boundary. It is assumed that the wave can be propagated, which implies that $\omega\Lambda^{\alpha} / c\Gamma(1+\alpha) > 1/2$. At the boundary we must have $E_1 = E_2$ and $(D^{\alpha}_{z} E_1) = (D^{\alpha}_{z} E_2)$, whence:

$$\begin{aligned} R &= 1 - A\left(\Lambda^{\alpha}\right)^{1/2 - i\beta}, \\ A &= \frac{2\left(\Lambda^{\alpha}\right)^{-1/2 + i\beta}}{1 + i\dfrac{c\Gamma(1+\alpha)}{\omega\Lambda^{\alpha}}\left(\dfrac{1}{2} - i\beta\right)}, \\ \beta &= \sqrt{\left(\frac{\omega\Lambda^{\alpha}}{c\Gamma(1+\alpha)}\right)^2 - \frac{1}{4}} \quad . \end{aligned} \tag{37}$$

IV. DISCUSSION

Figure 2 illustrates the behavior of the analytical solution (34) and (36) for the electric field $E(z)$ when α=0.92, $(\omega/c) = (5/\ \Lambda)^{\alpha}$ and Λ=12.5, we have also shown the analytical solution (18) and (19) for the electric field $E(z)$ when α=1, $(\omega/c) = 5/\ \Lambda$ and Λ=12.5.

From these results we can observe that one of the principal effects of considering the fractional order wave propagation in inhomogeneous medium is the appearance of an entire family of solutions (36) as a function of the fractional order parameter α. Also from figure 2 we notice that the wave length of the solution increases as the fractional order parameter α varies from 1 and approaches to 0. Additionally it can be noted that the solution obtained in (36) reduce to the previously known solution (19) for the limit case α=1, that has been previously reported in the literature [14], in this way the solutions obtained by the fractional calculus techniques are more general and contain as a limit case the well known solution for integer order wave propagation phenomena. It should be noted that the analytical results (34) and (36) are in good agreement with the approximated solutions previously obtained by Mohyud-Din *et al*. [20], where they have applied the homotopy analysis method to the wave-like fractional non-linear equation.

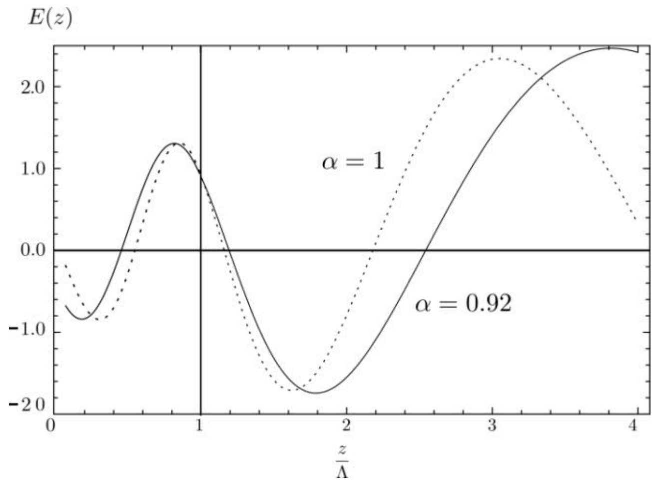

FIGURE 2. The analytical solution for the electric field $E(z)$ when α=0.92, $(\omega/c) = (5/\ \Lambda)^{\alpha}$ and Λ=12.5 (solid line) and α=1 (dashed line).

V. CONCLUSIONS

The analytical solutions for the fractional wave equation for an inhomogeneous medium have been obtained by considering a dielectric profile with a polynomial form $\varepsilon(z) = (a\ /\ (b+z^{\alpha}))^2$. We have illustrated these solutions and compared them with the results found for the integer wave equation for an inhomogeneous medium considering a dielectric profile with a polynomial form $\varepsilon(z) = (a\ /\ (b+z))^2$. The effect to varying the order of the space-fractional derivatives on the behavior of solutions has been investigated. We have noticed that for the fractional order case an entire new family of solutions $E(z)$ appears and the wave length of these solutions increases as the fractional order parameter α approaches to zero. We have shown through a simple example of wave propagation in inhomogeneous medium, the importance of introducing fractional order differential equations in physics and the necessity of introducing the fractional calculus techniques to the physics students.

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