Dynamics on a spherical surface in accelerated dilation with a pressure difference

Adrián G. Cornejo
Electronics and Communications Engineering from Universidad Iberoamericana, Calle Santa Rosa 719, C. P. 76138, Col. Santa Mónica, Querétaro, Querétaro, Mexico.

E-mail: adriang.cornejo@gmail.com

(Received 7 November 2013, accepted 25 February 2014)

Abstract
Dynamics on a spherical surface in accelerated dilation with a pressure difference is analyzed, where some regions of the surface may have one of three states states regarding to its deformation, which may remain flat, may be curved or perforated. When the surface is considered as a fluid, those three states can be described by the Bernoulli's equation.

Keywords: Pressure difference, Surface tension, Bernoulli's equation, Non-inertial frame of reference.

Resumen
Se analiza la dinámica sobre una superficie esférica en dilatación acelerada con una diferencia de presión, donde algunas regiones de la superficie podrían tener uno de los tres estados relacionados con su deformación, siendo que podría mantenerse plana, curvarse o perforarse, cambiando su forma. Cuando la superficie es considerada un fluido, esos tres estados pueden ser descritos por la ecuación de Bernoulli.

Palabras clave: Diferencia de presión, Tensión superficial, Ecuación de Bernoulli, Marco de referencia no inercial.

PACS: 01.55.+b, 47.55.dd,61.20.Gy, 64.60.Bd **ISSN 1870-9095**

I. INTRODUCTION

In the study about surfaces, spherical surface is mainly analyzed as a bubble related with the superficial tension and the pressure difference [1]. We can find that some region of a surface may have one of three states regarding to its deformation, which may remain flat, may be curved or may be perforated. Surface can be curved or distorted, for instance, by supporting some high quantity of matter, having the following scenarios,

a) No distorted surface, that supports nil or very low density of matter (only the pressure difference on the surface is noticed).

b) Curved surface by supporting some considerable density of matter. Also, objects about the density of matter will follow the curvature in their motion.

c) Curved and perforated surface by supporting a high density of matter and increasing of temperature, where the matter and the surface itself are draws (as a fluid) through the formed hole.

Assuming a spherical surface in accelerated dilation like a fluid, we find that each state of the surface can be described by the terms in the Bernoulli's equation.

In classical mechanics, total energy of a system includes different aspects of energy [2], defined as

$$E_T = PE_e + PE_g + KE, \quad (1)$$

where E_T is the total energy, P is the pressure, E_e is the elastic energy, E_g is the gravitational energy and KE is the kinetic energy. Total energy is commonly written as

$$E_T = \tfrac{1}{2}kx^2 + mgh + \tfrac{1}{2}mv^2, \quad (2)$$

where k is the elasticity constant, m is the mass, g is the constant of gravity for the Earth, h is the high and v is the velocity.

For a fluid system, an analogous expression is given according to the type of energy, having

$$E_F = PE_e + PE_g + KE, \quad (3)$$

where E_F is the fluid energy. This relation is described by the Bernoulli's equation [3] that relates pressure and velocity, defined as

$$p_1 + \rho g h_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \rho g h_2 + \tfrac{1}{2}\rho v_2^2, \quad (4)$$

where p is the pressure and ρ is the fluid density.

In this work, dynamics on a spherical surface in accelerated dilation with a pressure difference is analyzed for each one of the three states of the surface, which may remain flat, may be curved or perforated. When the surface is considered as a fluid, those three states can be described by the Bernoulli's equation.

II. NO DISTORTED SPHERICAL SURFACE IN DILATION WITH A PRESURE DIFFERENCE

Let us consider a homogenous spherical surface given by $A = 4\pi R^2$ in radial accelerated dilation from a central point, where R is the radius of the sphere. Spherical surface in dilation sweeps out the space during its dilation changing proportional to the increasing of radius. Then, changing of the surface multiplied by the acceleration, for a spherical surface in accelerated dilation, yields

$$adA = a4\pi dR^2, \tag{5}$$

where a is the radial acceleration. Considering that an interior force F_1 homogenously push out the spherical surface to be in accelerated dilated, we have the similar case of a spherical bubble with a surface tension γ [1] in dilation by effect of the interior force. When a force F_1 is applied from within the spherical surface exerting a pressure p_1, the surface tension is given by

$$F_1 = (2\pi R)\gamma \therefore \gamma = \frac{F_1}{2\pi R}, \tag{6}$$

where $2\pi R$ is the perimeter of its circumference. If now we assume that an external force F_2 is exerting a pressure p_2 from the external side of the spherical surface (for instance, due to the pressure exerted by an exterior bubble in contraction, as shown in Fig. 4), such force may cause a pressure difference $\Delta p = p_2 - p_1$ on such a surface, having

$$F_2 = \Delta p \pi R^2 \therefore \Delta p = \frac{F_2}{\pi R^2}. \tag{7}$$

Equating the forces, $F_1 = F_2$ from the expressions (6) and (7) results the so-called Laplàce's law [4], where considering that the spherical surface is only one film, is defined as

$$\Delta p = p_2 - p_1 = \frac{2\gamma}{R}. \tag{8}$$

In this case, surface is not distorted and nor perforated, then only pressure on the surface is present. Considering the surface as fluid, then Bernoulli's equation (4) is reduced to

$$p_1 = p_2. \tag{9}$$

Difference of pressure between the inner and outer sides of a bubble depends of the superficial tension, and it decreases with increasing radius of the spherical surface. If no force or a very few force acts normal to the surface, it must remain flat.

For the case of a spherical surface in accelerated dilation, radius R accelerated increases and the pressure difference Δp existent between both, inner and outer media is negative, where $p_1 > p_2$. In this way, force in the sphere in dilatation is greater than the force exercised from the external media.

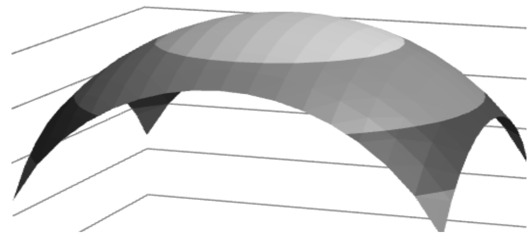

FIGURE 1. Section of a no-distorted spherical surface in dilation.

III. CURVED SURFACE BY THE DENSITY OF MATTER

Eventually, surfaces have to support some quantity of matter, which may distort the surface in dilation if the force exerted by the matter overpass the surface tension in a given point.

Let us now consider a spherical surface in accelerated dilation which can be elastically distorted by apply a force on it, for instance due to a considerable density of material that forms a body on the surface. So, considering a given amount of matter m on the surface that exerts a force, equaling expressions (7) and (8) and applying the second Newton law, yields

$$\gamma = \frac{F_2}{2\pi R} = \frac{ma}{2\pi R} \therefore R = \frac{ma}{2\pi\gamma}. \tag{10}$$

Then, changing of radius by the deformation directly depends of the quantity of matter in the body.

From expression (6), force is given by

$$F_1 = 2\pi R\gamma = ma. \tag{11}$$

From expression (8), we can write equivalence for a massive body, giving

$$\frac{\Delta pR}{M\rho} = \frac{\Delta pR}{V\rho^2} = \frac{2\gamma}{V\rho^2} \therefore \Delta p = \frac{2\gamma M}{V\rho R}, \tag{12}$$

where M is a massive body, ρ is the density of mater and V is the volume.

Magnitude of the surface distortion is measured by the changing in R at the distorted region.

If the pressure on one side of the surface differs from pressure on the other side, the pressure difference times surface area results in a normal force. In order for the surface tension forces to cancel the force due to pressure, the surface must be curved.

Surface curvature of a tiny patch of surface leads to a net component of surface tension forces acting normal to the center of the patch. When all the forces are balanced, the resulting equation is known as the Young–Laplace equation (8) [5].

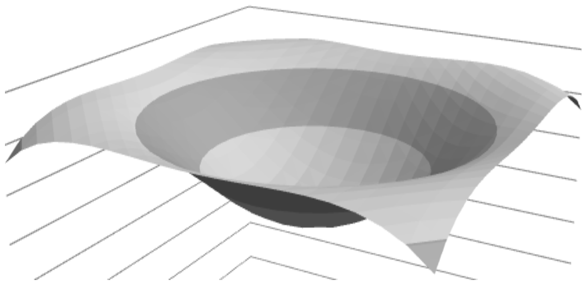

FIGURE 2. Section of a distorted spherical surface in dilation.

Furthermore, for a distorted spherical surface in accelerated dilation, matter about the distorted region will be moving on the geometry following the shape of surface. We can analyze path of the relative motion of a body *m* in the region of a distorted surface in dilation due to a high density of matter of other body *M*. Relative motion between the two bodies can be described by a non-inertial frame of reference that is traditionally derived by a coordinate transformation.

Let us consider a given body with mass *m* in circular motion with constant velocity *v* and radius *r* circumgyrating around a central point *O* on the *x* and *y*-axes [6]. Position vector is given by

$$r' = v_t t, \tag{13}$$

where v_t is the tangential velocity of the given body and *t* is the time.

If such a body in circular motion is also uniformly accelerated towards the vertical direction (it is, along the *z*-axis), then its position vector is given by

$$r_0 = v_0 t + \tfrac{1}{2} a t^2, \tag{14}$$

where v_0 is the initial velocity of the given body and *a* is its acceleration along that *z*-axis. Having that relative velocity is the velocity of a body (or a frame of reference) with respect to other; it is related only in systems of two bodies (or two frames of reference). Thus, relation between both, position in a fixed coordinate system and positions in the accelerated system, for a fixed observer is given by

$$r = r' + r_0 = v_t t + \left(v_0 t + \tfrac{1}{2} a t^2\right), \tag{15}$$

where its components in a three-dimensional frame of reference are given by

$$\begin{cases} r = v_t t \therefore t = \dfrac{r}{v_t}, \\ z = v_0 t + \tfrac{1}{2} a t^2. \end{cases} \tag{16}$$

Finding out time from the first expression in (16) and replacing it in the second expression of it, we have the equation of its trajectory as it is seen by a fixed observer on the given body, hence

$$z = v_0 \left(\frac{r}{v_t}\right) + \tfrac{1}{2} a \left(\frac{r}{v_t}\right)^2, \tag{17}$$

which is a parabola. If that acceleration starts from the rest, then initial velocity equals zero and expression (17) is reduced, giving

$$z = \frac{ar^2}{2v_t^2}. \tag{18}$$

We can generalize expression (18) for a spherical scenario extending vertical acceleration from along only one *z*-axis to several radial “*z*-axes” starting each one of them from a common central point [7]. Then, in a homogeneous radial acceleration, a sphere in accelerated dilation is formed. Thus, equation of the radial movement will be equivalent to the radius *R* of the formed sphere, hence

$$R = \frac{ar^2}{2v_t^2}. \tag{19}$$

Reducing radius in both sides, and reordering, yields

$$ar = 2v_t^2. \tag{20}$$

In addition, considering the surface as fluid, having from the Bernoulli's equation (4), yields

$$\Delta p = \tfrac{1}{2}\rho v^2 = \tfrac{1}{2}\rho a r \therefore ar = \frac{2\Delta p}{\rho}. \tag{21}$$

Equaling both equivalences (20) and (21) and reordering, yields

$$ar = 2v_t^2 = \frac{2\Delta p}{\rho} \therefore \Delta p = \rho v_t^2 = \rho a r. \tag{22}$$

Having that $r = h_1 - h_2$ represents the difference of distances between two points, and considering acceleration like the gravity $a = g$, we can write as

$$p_2 - p_1 = \rho g (h_1 - h_2). \tag{23}$$

Then, in this case Bernoulli's equation (4) is reduced to

$$p_1 + \rho g h_1 = p_2 + \rho g h_2. \quad (24)$$

IV. PERFORATED SURFACE BY A HIGH DENSITY OF MATTER

A surface can be perforated in a section where the superficial tension is zero. For instance, increasing the amount of matter in a body which is on a spherical surface in accelerated dilation, surface will increase its distortion. Such a density of matter could not perforate the surface until reach a certain density of matter that exceeds the threshold of the surface tension. Also considering that temperature of the center of mass increases its temperature, eventually the surface does not support more and it overcomes separating the particles that compose the surface being perforated in the expiration point, which is when the surface tension is zero [8, 9], forming a hole or singularity with section S (Fig. 4). In this case, matter is draining trough the hole as a flow due to the pressure difference, and Bernoulli's equation (4) is reduced to

$$p_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \tfrac{1}{2}\rho v_2^2, \quad (25)$$

where ρ is the average of density of matter of the fluid. For the average fluid velocity v, hence

$$\Delta p = p_2 - p_1 = \tfrac{1}{2}\rho\left(v_1^2 - v_2^2\right) = \tfrac{1}{2}\rho v^2, \quad (26)$$

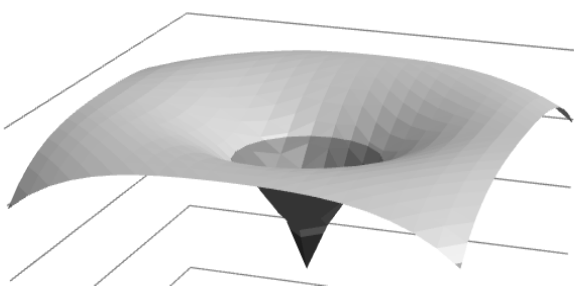

FIGURE 3. Section of a perforated spherical surface in dilation

V. DRAINING FROM A BODY TO OTHER BY THE PRESSURE DIFFERENCE

Let us consider a spherical surface with radius R_B in contraction, where a second spherical surface in dilation with radius R_A is on the spherical surface with radios R_B, where $R_A < R_B$, as shown in Fig. 4. When the pressure difference given by expression (8) is negative, it is when $p_1 > p_2$, and the surface is deformed by the high density of matter at a point of such a surface.

Considering the surface as fluid, a hole on the surface behaviors like a vortex with the vertex that coincides with singularity, where matter in the body m_1 with radius $R_1$is drained as a fluid to the body m_2 with radius R_2.

Furthermore, from the fluid dynamics, we have that the case of two spherical surfaces, one with a radius R_1 under the internal pressure p_1, and another with radius R_2 under the external pressure p_2, being connected by a pipe which put in contacts both spheres in communicating, the content of one of them pass to the other due to the difference of pressure between both media. Assuming that $p_1 > p_2$, it is for a negative pressure difference, we have that the content (for instance, a particle with charge q_1) in the spherical body m_1 with radius R_1 is drained towards the body m_2 with radius R_2 through the communicating tube.

The pressure difference between the spheres of radius R_1 and R_2 is given by

$$\Delta p = 2\gamma\left(\frac{1}{R_2} - \frac{1}{R_1}\right) = \tfrac{1}{2}\rho v^2, \quad (27)$$

thus, finding out for the square of the velocity, yields

$$v^2 = \frac{4\gamma}{\rho}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) = \frac{2\Delta p}{\rho}. \quad (28)$$

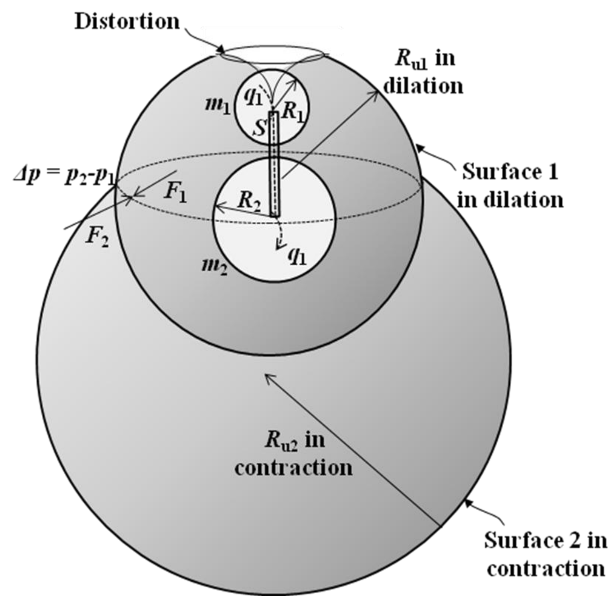

FIGURE 4. Difference of pressure between two bubbles that contain two bodies in contact through a tube like a singularity between them.

As result of the pressure difference, the density of matter ρ_1 within the sphere of radius R_1 flows through to the S cross-section tube that connects both spheres, with a rate given by the Bernoulli's theorem. Contents of volume V_1 in the sphere of radius R_1 is drained in a time dt to the volume V_2. The volume V_2 of the sphere of radius R_2 increases while the volume V_1 of the sphere of radius R_1 decreases, at a speed v according to

$$dV_1 = A_1 dR_1 = 4\pi R_1^2 dR_1 = Svdt, \quad (29)$$

then,

$$4\pi R_1^2 dR_1 = S\sqrt{\frac{4\gamma\left(\frac{1}{R_2}-\frac{1}{R_1}\right)}{\rho}}dt. \quad (30)$$

Integrating expression (30) and solving the integral by numerical procedures, yields

$$\int_{R_{01}}^{R_1}\frac{x^2dx}{\sqrt{(V-x^3)^{-1/3}-1/x}} = \frac{r^2}{4}\sqrt{\frac{4\gamma}{\rho}}t, \quad (31)$$

and volume is given by

$$V = R_1^3 + R_2^3. \quad (32)$$

If a closed system is considered, then the density of total matter in both volumes not change when it is moving from one sphere to the other one (incompressible flow), and the total volume of the system is preserved, so that the sum of the volumes of both bodies is constant, where $R_1^3 + R_2^3 = K$. Knowing the initial radius of the sphere R_2 is calculated at time t when this sphere comes within $R_2 \leq R_1 < V^{1/3}$.

Due to the perforation with section S, mass of radius R_1 behaves according to the Bernoulli's equation (4), where a particle velocity must to overcome the drained velocity (as an escape velocity) in order to out from the body avoiding to be drained. Thus, from the expression (26), square of the escape velocity for any particle which forms part of the spherical body of radius R_1, will be equivalent to the escape velocity of a mass body, hence

$$v_e^2 R_1 = \frac{2\Delta p R_1}{\rho_1} = \frac{2(2\gamma)}{\rho_1} \therefore R_1 = \frac{2\Delta p R_1}{v_e^2\rho_1} = \frac{4\gamma}{v_e^2\rho_1}, \quad (33)$$

where v_e is the escape velocity of the matter within the body with radius R_1.

VI. CONCLUSIONS

A spherical surface in a pressure difference may be distorted, then having a specific shape and dynamics. Distortion can be related with the quantity of matter on the surface. Thus, when the quantity of matter is very few (like some particles) and the superficial tension supports such a quantity of matter, the surface remains flat. On the other hand, surface could be deformed by a considerable quantity of matter. In addition, if the quantity of matter considerably increases, surface could be extremely deformed and eventually perforated in a section wherein the matter is drained as a fluid.

Furthermore, free matter about the distorted surface will change their trajectory following the changing of geometry of the surface in dilation, developing a determinate dynamics in the distorted system.

Regarding to the education, dynamics for surfaces and fluids are revisited describing the changes in the spherical surface in dilation, where it is showed the possibility to apply some of the known equivalences to consider another possible results and properties from the classical theories.

ACKNOWLEDGEMENTS

The author would like to thank Professor Sergio S. Cornejo for the review and comments for this paper.

REFERENCES

[1] Weinhaus F., Barker W., *On the equilibrium states of interconnected bubbles o balloons*, Am. J. Phys. **46**, 978-982 (1978).

[2] Goldstein, H., *Classical Mechanics, 2nd ed*. (Addison-Wesley Publishing Company, Inc. Reading, Massachusetts, 1980), pp. 91-103.

[3] Chanson, H., *Applied Hydrodynamics: An Introduction to Ideal and Real Fluid Flows*, (CRC Press, Taylor & Francis Group 2009).

[4] Pellicer, J., García-Morales, V., Hernández, M. J., *On the demostration of Young-Laplace equations in introductory courses*, Phys. Educ. **35**, 126-129 (2000).

[5] De Gennes, P., Brochard-Wyart, F., Quéré, D., *Capillary and Wetting Phenomena—Drops, Bubbles, Pearls, Waves*, (Alex Reisinger, Springer, The Nether Lands, 2002).

[6] Tocaci, E., *Relativistic Mechanics, Time, and Inertia,* (C. W. Kilmister Ed., Cambridge University Press, Cambridge, UK, 1984), pp. 183-251.

[7] Cornejo, A. G., *The gravity effect hypothesis*, Lat. Am. J. Phys. Educ. **5**, 697-699 (2011).

[8] Van Eijck, M. A., O' Connor, D., Stephens, C. R., *Critical Temperature and Amplitude Ratios from a Finite-Temperature Renormalization Group,* (2008) [arXiv:hep-th/9411226v1]

[9] Parola, A., Pini, D., Reatto, L., *Differential theory of fluids below the critical temperature: Study of the Lennard-Jones fluid and of a model of C_{60}*, The American Physical Society, (1995) pp. 2644–2656,
[DOI: 10.1103/PhysRevE.52.2644].