Magnetic field of an electric quadrupole moment in non-relativistic motion

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Abstract
We calculate the magnetic field of an electric quadrupole moment, and identify the magnetic quadrupole moment, in slow motion relative to an inertial reference system.

Keywords: Electric and magnetic moments, relative motion.

Resumen
Calculamos el campo magnético producido por un cuadripolo eléctrico que se mueve con velocidad no relativista para un sistema de referencia inercial e identificamos el momento cuadripolar magnético.

Palabras clave: Momentos eléctrico y magnético, movimiento relativo.

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Consider an electric quadrupole moment in rest, located at **r'** = $\mathbf{r}_0$(t). The associated electric dipole moment density (dipole moment per unit volume) is given by [1]

$$\wp'^i = -\frac{1}{6}\frac{\partial}{\partial x'_j}\mathbb{Q}'^{ij}, \tag{1}$$

in terms of the tensor electric quadrupole density $\mathbb{Q}'^{ij} = \mathbb{Q}_0^{ij}\delta^3(r'-r_0)$. It is assumed here that the net charge and net dipole moment vanishes, however there is a charge density [2]

$$\rho'^{(r',t)} = -\frac{\partial}{\partial x'_i}\wp'^i\,. \tag{2}$$

In a reference system where the quadrupole is moving with low velocity $\mathbf{v} = \mathrm{d}\mathbf{r}_0/\mathrm{dt}$, a current density is measured as [3]

$$\boldsymbol{j} = \rho\mathbf{v} = \rho'\mathbf{v},$$

$$= -\frac{\partial}{\partial x'_i}\wp'^i\mathbf{v}\,, \tag{3}$$

since at low relative velocities coordinates and velocities are the same. Eq. (3) can be written as the sum of two currents

$$\boldsymbol{j} = \boldsymbol{j}_m + \boldsymbol{j}_{\boldsymbol{p}},$$

where

$$\boldsymbol{j}_{\boldsymbol{m}} = -\nabla\times(\mathbf{v}\times\wp), \tag{4}$$

and

$$\boldsymbol{j}_{\boldsymbol{p}} = -(\mathbf{v}\cdot\boldsymbol{\nabla})\wp. \tag{5}$$

First, we work with (4) in order to identify the magnetic quadrupole moment $\boldsymbol{\mathcal{M}}$**.** Using the vector identity

$$\mathbf{a}\times\mathbf{b} = \varepsilon_{ijk}a^i b^j,$$

where ε^{ijk} is the Levi-Civita symbol, and we have assumed the convention sum over repeated indexes, we can write

$$(\mathbf{v}\times\wp)_i = \varepsilon_{ijk}v^j\wp^k.$$

Then,

$$(\mathbf{v}\times\wp)_i = -\frac{1}{6}\varepsilon_{ijk}v^j\frac{\partial}{\partial x^l}(\mathbb{Q}_0^{lk}\delta(r'-r_0)),$$

$$= \frac{\partial}{\partial x^l}\Big(\mathcal{M}^{il}\delta(r'-r_0)\Big)\,. \tag{6}$$

Here

$$\mathcal{M}^{il} = -\frac{1}{6}\varepsilon_{ijk}v^j\mathbb{Q}_0^{lk}\,, \tag{7}$$

is the tensor magnetic quadrupole moment generated by the motion of the electric quadrupole moment, with v^l the components of the velocity. Substituting (7) in (4) we obtain

$$j_m^i = -\varepsilon^{ijk}\frac{\partial^2}{\partial x^j \partial x^l}(\mathcal{M}_{lk}\delta^3(r-r_0)). \quad (8)$$

Similarly, from (5) we obtain

$$j_p^i = \frac{1}{6}v_j \mathbb{Q}_0^{ik}\frac{\partial^2}{\partial x^j \partial x^k}(\delta^3(r-r_0)). \quad (9)$$

Next, we compute the vector potential due to (8),

$$A_m^i(r) = \frac{1}{c}\int \frac{j_m^i(r')}{|r-r'|}d^3r'.$$

The integration is performed using the well known property of Dirac´s delta

$$\int [\frac{\partial^n}{\partial x^n}\delta(x)]\varphi(x) = (-1)^n \int \delta(x)\left[\frac{\partial^n}{\partial x^n}\varphi(x)\right]dx,$$

then

$$A_m^i(r) = -\frac{1}{c}\varepsilon^{ijk}\mathcal{M}_{jk}\int \frac{\delta^3(r')}{|r-r'|^3}d^3r'$$

$$-\frac{3}{c}\varepsilon^{ijk}\mathcal{M}_{lk}\int \frac{(x_j - x_j{}')(x_l - x_l')}{|r-r'|^5}\delta^3(r')d^3r',$$

$$= \frac{1}{cr^5}\varepsilon^{ijk}\left(3x_j x_l - r^2\delta_{jl}\right)\mathcal{M}_{lk}\,, \quad (10)$$

which has the form of the vector potential due to a magnetic quadrupole moment [3]. The contribution from $\boldsymbol{j_p}$ to the vector potential is

$$A_p^i(r) = \frac{1}{6cr^5}\left[3(\mathbf{r}\cdot\mathbf{v})x_k - r^2 v_j\right]\mathbb{Q}^{ik}. \quad (11)$$

The magnetic field is calculated by

$$\boldsymbol{\mathcal{B}} = \nabla \times \boldsymbol{A}.$$

In this way, from (10) we have

$$\mathcal{B}_m^i(r) = \varepsilon^{ijk}\frac{\partial}{\partial x^j}A_{mk},$$

$$= -\frac{1}{cr^7}\varepsilon^{ijk}\varepsilon_{pqk}[r^2(3\delta_j^p x^l + 3\delta_j^l x^p - 2\delta^{lp}x_j)$$

$$-5x_j(3x^p x^l - r^2\delta^{lp})]\mathcal{M}^{lq}.$$

We use the result

$$\varepsilon^{ijk}\varepsilon_{klm} = \delta_l^i\delta_m^j - \delta_m^i\delta_l^j,$$

where δ_l^i is the Kronecker delta symbol, to obtain

$$\mathcal{B}_m^i(r) = \frac{3}{cr^7}[5\left(x_l\mathcal{M}^{lj}x_j\right)x^i$$

$$-r^2(\mathcal{M}^{ij}x_j + \mathcal{M}^{ji}x_j + \mathcal{M}_l^l x^i)], \quad (12)$$

which has the form of the magnetic field due to a magnetic quadrupole moment [3].

From (11) we obtain the corresponding magnetic field contribution

$$\mathcal{B}_p^i(r) = \frac{1}{cr^7}[r^2(-\mathcal{M}^{im}x_m + \frac{1}{2}\varepsilon^{ijk}x_j\mathbb{Q}_{kl}v^l)$$

$$-\frac{5}{6}\left(\varepsilon^{ijk}x_j\mathbb{Q}_{kl}x^l\right)(\boldsymbol{r}\cdot\mathbf{v})]. \quad (13)$$

Summing (12) and (13) gives the total magnetic field, which we have separated in these terms in order to reproduce known results in the multipole expansion of the vector potential and the magnetic field. We have defined the magnetic quadrupole moment [7] following the same definition for the magnetic dipole moment due to an electric dipole moment in slow motion as in [3]. This leads to the expression (12) for the magnetic field from a magnetic quadrupole moment.

VI. CONCLUSIONS

The conclusions must notice the new and remarkable contributions of the paper. Also the suggestions and shortcomings of the manuscript must be pointed out.

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REFERENCES

[1] Heras, J. A., *Radiation by moving quadrupoles*, (Instituto de Física, Universidad Nacional Autónoma de México, 2000).
[2] Raab, R. E. and de Lange, O. L., *Multipole theory in Electromagnetism, Chap.* 1 (Clarendon Press-Oxford, UK, 2005).
[3] Hnizdo, V., *Magnetic dipole moment of a moving electric dipole*, Am. J. Phys. **80**, 645-647 (2012).