Let us derive Lorentz Transformation Equations through a more heuristic way for students

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Abstract

The Lorentz Transformation Equations (LTEs) are being developed here merely through applying the time dilation and length contraction concepts. Here the emphasis is on the connection which must logically be there between the kinematic measurements made by the observers of the two different inertial systems. The key issue throughout the present derivation of LTEs is this expectation that different observers, though observing events differently, think and judge in a reasonable way. In this dialectical approach, the students' logical judgment is being invited actively.

Keywords: Special relativity; time dilation; length contraction.

Resumen

Las ecuaciones de transformación de Lorentz (LTEs) se están desarrollando aquí meramente a través de la aplicación de la dilatación del tiempo y de los conceptos de contracción de longitud. Aquí el énfasis está en la conexión que debe lógicamente estar allí entre las mediciones cinemáticas realizadas por los observadores de los dos sistemas inerciales diferentes. La cuestión clave en toda la presente derivación de LTE es la expectativa de que, diferentes observadores, aunque observando los acontecimientos de otra manera, de pensar y de juzgar de una manera razonable. En este enfoque dialéctico, se invitó a juicio lógico de los estudiantes de forma activa.

Palabras clave: Relatividad especial, dilatación del tiempo, contracción de longitud.

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I. INTRODUCTION

The Lorentz Transformation Equations (LTEs) are derived in numerous textbooks ([1, 2, 3], e.g.). The derivation procedure there generally begins with introducing the time dilation and then the length contraction concepts. Hence, after showing that the LTEs must be linear, the unknown coefficients are determined one by one.

The problem with such derivations is that the reader is suddenly faced there with the final form of LTEs, without a feeling of having a stake in developing these mysterious equations. Students, when offered these kinds of derivations for the first time, feel checkmated by pure mathematics.

The standard derivation of LTEs is better to be supplemented with a more heuristic approach towards them, in which, as explained here, the student's assistance and judgment is being invited. Of course Ugarov [4] and Mould [5] developed a heuristic approach towards LTEs. But the former lacks sufficient argument, while the latter, though profound, sounds somewhat complicated and lengthy. In the approach presented here, the emphasis will be on a direct and clear dialogue. I have found that this approach helps students to assimilate special relativity quite naturally, instead of perpetually recognizing it as a subject entirely in contrast with common sense. In the next section, the Lorentz Transformation Equations are derived in this manner.

II. DIALECTICAL ARGUMENT LEADING TO LTEs

In this derivation, it is assumed that the student has learnt the principles of special relativity (constancy of the speed of light and the legitimacy of all inertial observers) and the time dilation and length contraction concepts in consequence. He/she is also supposed to be taught the subtleties of simultaneity and, meanwhile, the possibility of synchronizing all clocks which are at rest with respect to each other. These are the common preliminaries to whatever approach towards LTEs.

Having these assumptions in mind, let's imagine two Cartesian coordinate systems S and S' which the latter is moving relative to the former at a velocity v along the x-axis (see Fig.1). The x-axis in S is aligned with the x'-axis in S', while (recalling the space symmetry properties) their respective y and y' axes, as well as their respective z and z' axes, remain parallel during their relative motion. The two clocks, located at O and O′, have been set on zero at the *event of coincidence*.

Now imagine a firework which is set off on the *x*-axis of system S (at rest) and also leaves a mark on the moving axis *x*'. Hereafter, we call that explosion "the event A", and show its space-time coordinates by (x, t) and (x', t') as measured by the observers at O and O$'$, respectively. The event A could have been imagined to occur off the *x* and *x*' axes, but for the sake of convenience (without losing the generality) we neglect this possibility in present.

To find the connection between these two sets of coordinates we will follow the subject as may be argued *by the observer at* O. The key issue throughout the present derivation of LTEs is this expectation that *the observer at* O *can predict and justify what the observer at* O' *will observe*. That issue means that "the relativity theory is *fully rational*".

However, each of these two observers considers their own measurements as the correct ones and those of the other observer as mistakes. But, instead of simply agreeing to differ, the different observers can finally understand why they record different measurements. Put another way, *different observers, though observing events differently, think and judge in a reasonable way*.

As shown in Fig.1, the observer at O would measure x as

$$x = vt + \overline{\mathrm{O'A}}\,, \qquad (1)$$

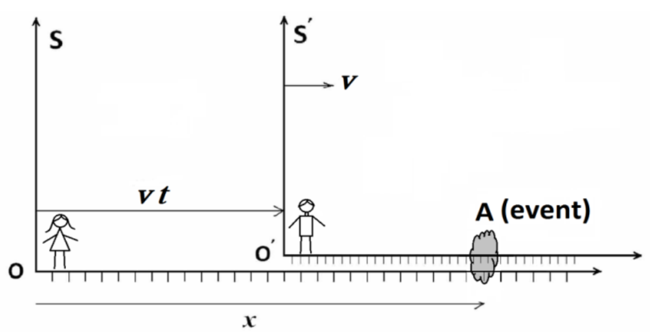

FIGURE 1. As seen by the observer at $\boldsymbol{O}$, the event A (the explosion of the firework) happens at time *t*. She also sees that the length marks on the meter-stick attached on the *x*'-axis of S' system are contracted by the factor $\boldsymbol{1/\gamma = (1 - v^2/c^2)^{1/2}}$. Thus she reasons that the observer at $\boldsymbol{O'}$ must assign a "wrong" value $\boldsymbol{x' = \gamma \times \overline{O'A}}$ to the length of the segment $\boldsymbol{\overline{O'A}}$. The observer at $\boldsymbol{O'}$, logically, must assign the same value to the location of the explosion is his own system S'.

where vt is the distance of O$'$ from O at time$t$. The same letter "A" is used above to show the location of event A. On the other hand, the observer at O already knows (and sees) that the meter-stick attached on the *x*'-axis of S' has been contracted by the factor $1/\gamma = (1 - v^2/c^2)^{1/2}$. That is the length marks on *x*'-axis would seem to O to be contracted by the factor $1/\gamma$.

Therefore, the observer at O would conclude that: "*The observer at* O' *must assign the "apparent" location* $\mathbf{x}' = \gamma \times \overline{O'A}$ *to* A" (see Fig. 1). Note that the S$'$ observers can measure $\mathbf{x}'$ directly (the firework has left a mark on the x'-axis) and, logically, they *must* obtain the very value for the location of the explosion. So, we can put $\overline{\mathrm{O'A}} = x'/\gamma$ in Eq. (1) and obtain $x = vt + x'/\gamma$ which, after rearranging for *x*', takes the desired form as

$$x' = \gamma(x - vt). \qquad (2)$$

That is the first LTE.

The last step will be the derivation of the second LTE which relates t' to *x* and *t*. The subject will be analyzed again by the observer located at O. As before, it is sensible to expect that if her reasoning was sound the observations at O$'$ would be the same as what she predicts to be.

The observer at O sees that when the event A happens (at the time *t*), the moving clock at O$'$ reads simultaneously a time t'_1, say. Namely, as to O, the explosion and the event of the O$'$ clock reading t'_1 are simultaneous. So, regarding the time dilation effect, t'_1 should be related to t as

$$t'_1 = t/\gamma\,. \qquad (3)$$

Note that the time dilation relation is applicable here because the two clocks have been synchronized at the event of their coincidence ($t = t' = 0$).

Obviously, the observer at O$'$ will not notice the event A at the very timet'_1, but instead only when he has received the light pulse propagated in space by the firework (see Fig. 2). As seen by O, that will take a time δt. However, the point O$'$ will move a distance $v\delta t$ rightward during that time. So, as seen by the observer at O, we have

$$c\delta t + v\delta t = \overline{\mathrm{O'A}}. \qquad (4)$$

Now, using equation (1) to substitute $\overline{\mathrm{O'A}} = x - vt$ in equation (4), and then solving it for δt, we obtain

$$\delta t = \frac{x - vt}{c + v}\,. \qquad (5)$$

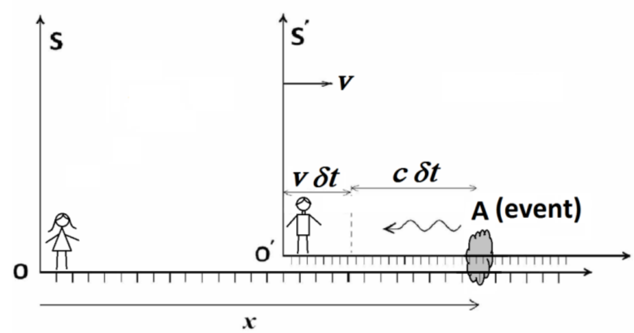

FIGURE 2. As seen by the observer at O, the pulse emitted in (the event) A at *t* travels a length $c\delta t$ to arrive at O$'$ at $t + \delta t$. By that time, O$'$ itself has moved a length $v\delta t$ rightward.

This way, the observer at O would assign a time $t + \delta t$ to the "event B", *i.e.* the event of *the observer at* O' *receiving the pulse.*

Meanwhile, according to the time dilation effect, she must also see that the clock at O′ reads $(t+\delta t)\,/\gamma \equiv t'_2$ at the event B. The observer at O′ must (logically) be in agreement with O in that he received the pulse when his clock read the time $t'_2 = (t+\delta t)\,/\,\gamma$ (events cannot be denied). But he would not misinterpret that time as the happening moment of A, because he is aware of the time that the pulse has taken to travel the distance from point A (*i.e.* the mark left by the firework on x′ axis) to O′. So, instead, he must reason: "*the firework was set off at x', and the light pulse has been moving toward me with a speed of c; so the pulse must have been in the way for a time x'/c before I received it. Thus the true happening time of A should be calculated by taking x'/c away from t'_2*". Thus

$$t' = t'_2 - \frac{x'}{c} = (t+\delta t)\,/\,\gamma - x'/c. \qquad (6)$$

The substitution of x' from Eq. (2), and δt from Eq. (5), into Eq. (6) will result in

$$t' = \frac{1}{\gamma}\left(t + \frac{x-vt}{c+v}\right) - \frac{\gamma(x-vt)}{c}, \qquad (7)$$

which after some algebra takes the desired form of

$$t' = \gamma(t - vx/c^2)\,, \qquad (8)$$

as the second Lorentz Transformation Equation.

It will be straight forward to generalize the above argument to the situation where the event A also assumes lateral coordinates. There, according to the symmetries of the Euclidean space, the two simple equations $y = y'$ and $z = z'$ will be added to LTEs, while equations (2) and (8) remain unchanged.

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