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Modeling transverse train seat occupation with one-dimensional potential barriers

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Abstract

Occupation of a six-seat row in a train is described using potential barriers. Assuming that the row is filled one seat at a time, a seating model is constructed using seat occupation probabilities based on a seat's ability to preserve passenger privacy. Introducing potentials associated with these probabilities for each seat on the row gives rise to a symmetric, multi-step potential barrier. Transmission and tunneling through these barriers are then used to describe likelihood of moving to adjacent or non-adjacent seats.

Keywords: Mathematical modeling, quantum mechanics, potential wells

Resumen

Se describe la ocupación de una fila de seis plazas en un tren, utilizando potenciales de barrera. Suponiendo que en la fila está ocupado un asiento a la vez, se construye un modelo de asiento utilizando las probabilidades de ocupación de asiento, con base a la capacidad del asiento para preservar la privacidad de los pasajeros. La introducción de los potenciales asociados con estas probabilidades para cada asiento en la fila, da lugar a un potencial de barrera simétrico de varios pasos. La transmisión y el túnel a través de estas barreras se utilizan para describir la probabilidad de pasar a los asientos adyacentes o no adyacentes.

Palabras clave: Modelación matemática, Mecánica cuántica, Pozos de potencial.

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I. INTRODUCTION

As a mathematical theory, quantum mechanics is elegant and eloquent. Unfortunately, the machinery required to attain this comes at a price. Conceptual difficulties concerning energy and tunneling through potential wells, for example, have been reported [1]. This may not be surprising given the non-intuitive concepts and abstraction of quantum theory.

However, just like counter-intuitive problems, non-intuitive concepts prevent trivialization and promote critical thinking [2].

In recent years, conceptual difficulties have been addressed with moderate success using modelling [3] and mathematical models. The latter is well known and the literature is replete with its use in many fields, not just in Physics.

One important aspect of conceptual and mathematical models is the paradigm shift, and quite possibly attitude shift, on and towards concepts that it can provide. For example, it may be beneficial for some, to see concepts applied in other ways or means from those adopted in practice. These alternative scenarios may explain some points more clearly and at times may provide new insight. Either way the chance of attaining a thorough understanding is improved.

In this paper, we present a model that applies the concepts of quantum mechanics to the familiar everyday experience of changing seats on a train. Using potential wells, the probability of changing or migrating to an unoccupied seat is investigated. Some results are counterintuitive, resulting from the mathematics of quantum mechanics while some are intuitive and is described rather well by the model.

In the next section, an algorithm for train seat selection is presented with the aim of preserving the privacy of the passenger. This is followed by seat occupational probabilities and seat potentials based on seat popularity in Section III. The mathematics of step-up and step-down potentials is applied to passengers moving to adjacent seats in Section IV. The case of migration to non-adjacent seats is discussed briefly in Section V.

II. SEAT SELECTION

Consider a transverse row of six train seats, S_i where i = 1, 2, 3, 4, 5, 6, with S_1 and S_6 making up the ends of the row (see Fig. 1). A passenger's choice of seat is motivated by the desire to keep his or her privacy for the duration of the journey. This is accomplished when a passenger does not sit next to any other passenger on the row. A seating arrangement that achieves this goal will be called a configuration [4].

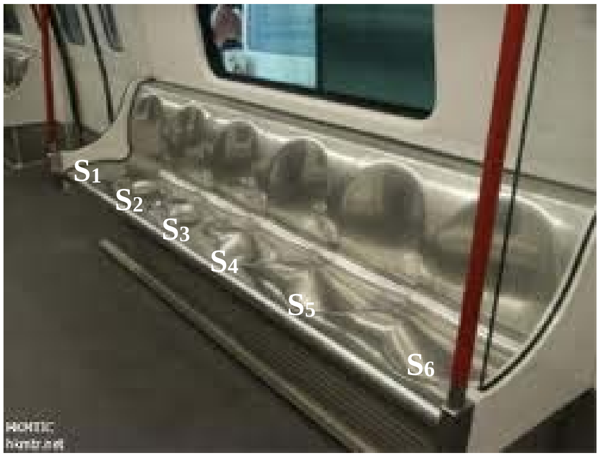

FIGURE 1. A row of six train seats in a transverse configuration [5]. In our example, the seats S_1 and S_6, adjacent to the Plexiglas panels at the ends of the row are the most desirable for passengers and will be occupied most of the time.

A configuration is saturated if the privacy of any seated passenger is violated when the next passenger sits down.

That is, in a saturated configuration, one or two seated passengers will lose their privacy when the next passenger sits down.

Let us consider a possible configuration, assuming that the row is filled one seat at a time. The first passenger can choose any seat but by choosing S_1 or S_6 he or she is guaranteed to sit next to one other passenger if the row is filled. This guarantee extends to the second passenger who occupies the end-of-row seat not chosen by the first passenger. By the time the third passenger choses, only four seats, S_2, S_3, S_4, and S_5, are available and the guarantee enjoyed by the first two passengers can no longer be enjoyed.

In order to maintain privacy the third passenger opts for either S_3 or S_4 because S_1 and S_6 are occupied.

These considerations suggest that saturation is attained after the third passenger sits down. Similar seating algorithms assume that the remaining seats will be chosen at random post-saturation [4].

In our example because the third passenger's choice between S_3 and S_4 depends on the first passenger's choice, we will assume for simplicity that the fourth passenger occupies either S_3 or S_4 depending on the third passenger's choice.

That is, the fourth passenger sits next to the third passenger but not next to the first passenger or the second passenger. This leaves S_2 and S_5 as the remaining seats to be filled. Note that occupants of these two seats sit next to two passengers instead of one.

III. OCCUPATIONAL PROBABILITIES

In the seat selection process presented, seats are chosen in order to preserve privacy. We will assume, given that the violation of privacy is imminent, that it is more preferable to sit next to one passenger than it is to sit next to two. Consequently some seats will be more desirable than others and these more desireable seats will be occupied most of the time. Thus the probability of more desirable seats being occupied has to be greater than that of less desirable seats.

Let $P(S_i)$ be the probability that the seat S_i is occupied.

If passengers must sit, then the sum of $P(S_i)$ taken over all seats on a row is unity. According to the model described in the previous section, S_1 and S_6 will have the greatest occupational probabilities because it is more desirable to sit next to one passenger than two (see Fig 2(a)).

We conclude that, $P(S_1) = P(S_6)$ since passengers occupying S_1 and S_6 are guaranteed to sit next to just one passenger. Some may choose seat 1 over seat 6, and vice versa, depending on the direction of travel for example, but this does not affect their occupational probabilities. Also from the previous section, we have $P(S_3) = P(S_4)$ whether S_3 or S_4 is chosen first and that $P(S_1) > P(S_3)$. The seats S_2 and S_5 have the lowest occupational probabilities and because occupants of these seats sit next to two passengers, we will take $P(S_2) = P(S_5)$ and $P(S_5) < P(S_3)$.

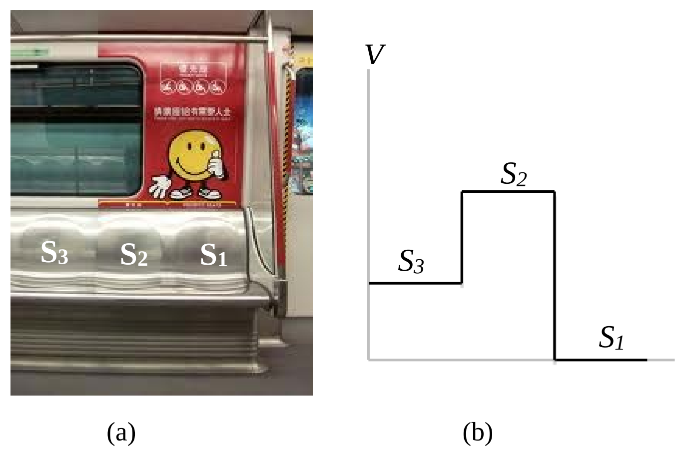

FIGURE 2. (a) Three seats in a transverse row [5]. (b) Corresponding seat potentials. Lower seat potential is associated with more desirable seats. The seat S_1 adjacent to the Plexiglas panels is the most desirable amongst the three shown here.

We represent occupational probabilities graphically using seat potentials as shown in Fig 2(b). In this representation, a potential well one seat wide is used to represent a seat along the row with the depth of the potential well indicating the occupational probability for the corresponding seat.

Thus the potential wells representing seats 1 and 6, 3 and 4, 2 and 5, have equal depths (see Fig. 2). In addition, the potential wells representing S_1 and S_6 are deeper than those represented by S_3 and S_4, which in turn, are deeper than those represented by S_2 and S_5.

In the next section, we will model a seated passenger as a particle in a corresponding potential well with one particle per well. A potential well is empty when the seat it represents is unoccupied.

IV. MOVING TO AN ADJACENT SEAT

An unoccupied seat, regardless of its occupational probability, may or may not be occupied. The vacancy can be filled either by a passenger wishing to sit on the row, or by another passenger already seated on the row. In the latter case, we will assume that the passenger is seated adjacent to the seat to be occupied. Considering adjacent seats only ignores other seats, occupied or unoccupied. This means that changing seats may violate a passenger's privacy. The associated seat potentials will be scaled in such a way to make the seat with greater occupational probability have zero potential (see Fig. 2 and Fig. 1). There is no loss in generality by using this procedure. Three possible outcomes of seat migration are discussed below.

A. Moving from a more popular seat

The first case is when a passenger moves from a seat with greater occupational probability; for example from S_1 to S_2 which is essentially a step-up potential (see Fig. 3 inset).

We see that a potential step of height V_0, serving as a demarcation between two adjacent seats with $V = 0$ for the region $x < 0$ representing a seat with greater occupational probability and $V = V_0$ for the region $x > 0$ representing a seat with lower occupational probability

We model a moving passenger as a beam of particles travelling from the region $x < 0$ to $x > 0$. The fact that each seat in the row has equal and finite width is not taken into account. In Eq. (1), the reflection amplitude coefficient r, assuming that the particle's energy $E > V_0$, is given by [6]:

$$r = \frac{p - \sqrt{p^2 - 2m(E - V_0)}}{p + \sqrt{p^2 - 2m(E - V_0)}}. \quad (1)$$

We interpret r as a quantity related to the probability of staying in the current seat. With this interpretation, for given values of momentum p, mass m, and energy $E > V_0$, greater V_0 yields greater r as shown in Fig.3.

This suggests that Eq. (1) indicates that it is less likely that a passenger in S_1 will move to S_2 compared to a passenger in S_4 moving to S_5. This is a reasonable result since, from Section 2, $P(2) = P(5)$ and $P(1) > P(4)$.

Furthermore, because $P(4) = P(3)$, then moving from S_3 to S_2 is just as probable as moving from S_4 to S_5. Thus a passenger is less likely to give up a seat that guarantees having to sit, at the most, to another passenger.

These considerations suggest that there is satisfactory agreement between the mathematics of potential wells and the changing of seats of passengers in a train.

B. Moving from a less popular seat

The second case is when a passenger moves from a seat with lower occupational probability; for example from S_5 to S_6 (see Fig .4). This is a step-down potential. Figure 4 illustrates a potential step of height V_0. This potential distinguishes between two adjacent seats with $V = V_0$ for the region $x < 0$ representing a seat with lower occupational probability and $V = 0$ for the region $x > 0$ representing a seat with higher occupational probability. As in the previous discussion, the finite width of the seats is not taken into account.

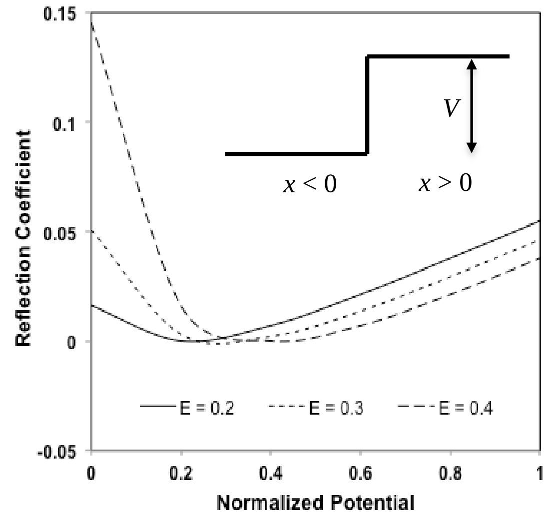

FIGURE 3. Reflection coefficients for a passenger moving from a more popular seat (low seat potential) to a less popular seat (high set potential). This is effectively a step-up potential. The general trend is consistent across different values of E. INSET: Step-up potential of height V_0.

A moving passenger is modeled as a beam of particles travelling from the region $x < 0$ to $x > 0$. The reflection amplitude coefficient r, assuming that the particle's energy $E > V_0$, is given by [6];

$$r = \frac{p - \sqrt{p^2 - 2m(E + V_0)}}{p + \sqrt{p^2 - 2m(E + V_0)}}. \quad (2)$$

We continue to interpret r as a quantity related to the probability of staying in the current seat. With this interpretation, for given values of momentum p, mass m, and energy $E > V_0$, greater V_0 yields greater r.

This result is counterituitive becuase the passenger is moving to a more popular seat, which is essentially a step-down potential (see Fig.4 inset). This means that there is a smaller chance of staying in the current seat, which is less popular. Furthermore, we find that greater V_0 increases r.

These counter-intuitive observations, described in the context of square potential wells as paradoxical reflection and paradoxical confinement, has been reported previously [7].

In Fig. 4, we compare the reflection coefficients for a passenger moving from a less popular seat and a passenger moving from a more popular seat according to Eq. (2) and Eq. (1) respectively. Surprisingly the probability of staying in an unpopular seat is greater than staying in a popular seat, verifying paradoxical reflection.

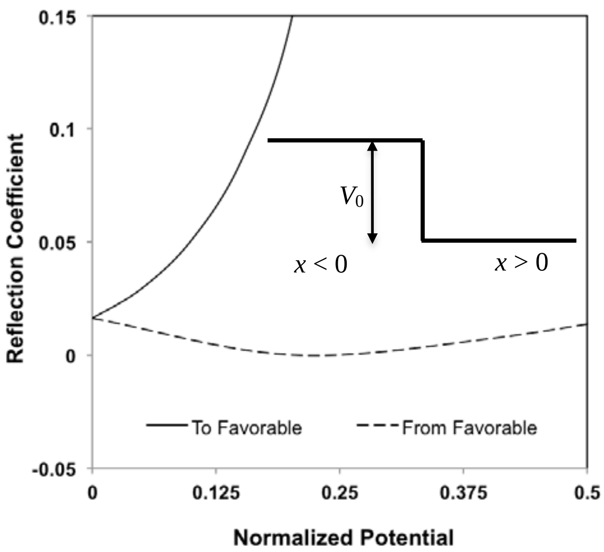

FIGURE 4. Comparing reflection coefficients for a passenger moving from a less popular seat (high seat potential) and a passenger moving from a more popular seat (low set potential) for E = 0. INSET: Step-down potential of height V_0.

We conclude that in this particular case, the mathematics of potential wells do not completely and satisfactorily describe passenger seat migration on trains.

C. Moving to an equally popular seat

The third case is when a passenger moves to a seat with equal occupational probability to the one currently occupied. The potential $V_0 = 0$ and this is only possible when moving from S_3 to S_4 or from S_4 to S_3 (see Fig. 2). Because $V_0 = 0$, using Eq. (1) or Eq. (2) gives $r = 0$, suggesting that a passenger will always move to a seat that is perceived to be as desirable as the current one occupied. Clearly, this is not a practical result, and should be interpreted in the context of our whole discussion.

V. MOVING TO A NON-ADJACENT SEAT

In the previous section, step potentials were used to describe passengers that move to adjacent seats. In this section, we will consider the case when a passenger moves to non-adjacent seats. We will ignore the seats that do not play a part in the migration and any possible violation to passengers' privacy.

A migrating passenger will be modelled as a particle that tunnels through intermediate seats. Again, the associated seat potentials will be scaled in such a way to make the seat with greater occupational probability have zero potential.

Consider a passenger that *tunnels* from a seat with greater occupational probability, for example from S_1 to S_3 through S_2, which is represented by a potential barrier of width L as shown in Fig. 5. Initially the passenger is in the region $x < 0$ representing a seat with greater occupational probability and tunnels a distance L to the region $x > L$ representing a seat with lower occupational probability.

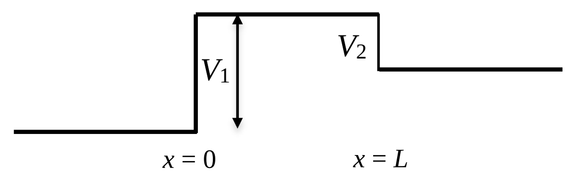

FIGURE 5. A potential barrier of width L. The region with lower potential represents a seat that is chosen more often compared to the seat represented by the region with higher potential.

We will base our discussion assuming that $V_1 = V_2$ which is customary in the literature. During the seat selection process the passengers choose the best available seat without making provisions for a possible change in seat. Because the best possible seat was chosen and occupied, the desire to change seats is taken to be minimal. Assuming that the particle's energy is low, the transmission amplitude coefficient T, is given by [8];

$$T \approx e^{-c\sqrt{V}L}. \tag{3}$$

where $V_1 = V_2 = V$ and c is a constant while the term $\sqrt{V}L$ approximates the area of the barrier. As a first approximation then, the transmission probability T is dependent on the area of the barrier and the greater the area, the lesser the chances of tunneling or moving seats. These considerations suggest, for example, that it is less likely that a passenger will move from S_1 to S_6 than moving from S_2 to S_5.

Using Eq.(3) as a coarse approximation when $V_1 \neq V_2$ suggest that it is less likely for a passenger to move from S_1 to S_4 than from S_1 to S_3, even if $P(3) = P(4)$, while the probability of moving from S_1 to S_5 is even lower. In addition, the symmetry in seat potentials of the presented model indicate that moving from S_6 to S_4, which is equally probable as moving from S_1 to S_3, is more likely than moving from S_6 to S_3.

We note from Eq. (1) and Eq. (2) that $r^2 < 1$, where r^2 is related to the probability of not moving to an adjacent seat and from Eq. (3) that $R + T \approx 1$, where R is related to the probability of not moving to a non-adjacent seat. Using these relations we have $R - r^2 > T$ which for V sufficiently high with $p^2 > 2m(E \pm V)$ leads to $R > r^2$.

These considerations imply that it is more likely that a passenger will move to an adjacent seat than a non-adjacent seat. This serves to set the bounds for the validity of the model, as this trend is not true for all vales of energy.

VI. CONCLUSIONS

This work presents a real-life, familiar, application of the mathematics used in quantum mechanics. It is hoped that this paper would help improve the conceptual understanding of potential wells and barriers by providing a different application but similar interpretation of the relevant mathematics. This paper also illustrates that the mathematics of quantum mechanics can be non-intuitive, a characteristic of most, if not all, concepts of quantum theory.

The model presented in this work assumes that train seat occupation probabilities do not change, and that when a passenger chooses a seat, there is no knowledge of subsequent passengers. Passengers also, need not fill the row sequentially as other rows or even cars are available.

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