A gentle introduction to *RC* and *LR* circuits

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Abstract
Two standard circuits in a first course in physics are of a resistor and a capacitor or inductor connected in series to a battery via a switch. The conventional approach to such circuits is to begin with the differential equations describing them or, at minimum, to write down the exponential functions for the solutions of those equations. However, student understanding is improved by initially focusing on conceptual ideas and basic algebra to discuss their time-dependent behavior.

Keywords: *RC* and *LR* circuits, Time constant, Kirchhoff's voltage loop rule.

Resumen
Dos circuitos estándar en un primer curso de física son de una resistencia y un condensador o inductor conectados en serie a una batería a través de un interruptor. El enfoque convencional de tales circuitos es comenzar con las ecuaciones diferenciales que los describen o, como mínimo, escribir las funciones exponenciales para las soluciones de esas ecuaciones. Sin embargo, la comprensión del estudiante mejora al enfocarse inicialmente en ideas conceptuales y álgebra básica para discutir su comportamiento dependiente del tiempo.

Palabras clave: Circuitos *RC* y *LR* constante de tiempo, regla de bucle de voltaje de Kirchhoff.

I. INTRODUCTION

A single-loop circuit consisting of a battery of emf ε, a switch S, a resistor R, and either a capacitor C or an inductor L are students' usual first introduction to time-dependent circuits [1] in which the current, the voltages across R and across C or L, and the charge on C or magnetic flux in L asymptotically approach their final values. Many textbooks jump into the mathematics, either by solving the differential equation obtained from a circuit analysis or by writing down its exponential solution without derivation. However, conceptual understanding is improved if class time is invested in explaining the ideas behind these series circuits using concepts and elementary algebra alone, before plunging into the details of exponentials and calculus.

II. SIMPLIFIED CIRCUIT ANALYSES

A. Series *RC*

Following the sequence of a standard first course in physics, consider the *RC* circuit illustrated in Fig. 1. The capacitor is initially uncharged (so that $q_0 = 0$) and a timer is started from zero at the instant the switch S is closed. After that instant, current I begins to flow, successively carrying small amounts of charge off the negative plate and onto the positive plate of the capacitor. It therefore requires some time τ before the magnitude of the charge q on either plate reaches a significant fraction of its final charge q_∞. As this last subscript suggests, it theoretically takes an infinite amount of time for the capacitor to fully charge up. The reason it takes a long time is that, as the charge on the right-hand plate of the capacitor in Fig. 1 approaches q_∞, this large positive charge repels any further increment of positive charge that the current tries to add to it. Another way of saying the same thing is to note that as the capacitor charges up, the voltage $V_C = q/C$ across it increases; the situation then becomes like that of connecting one battery of voltage ε to a second battery of voltage V_C with their positive terminals wired together so that they are competing to drive current in opposite directions. The larger of the two voltages will force the current to flow away from its positive terminal; but if the two voltages become equal to each other, then the current I will be zero. In other words, the current has its largest value I_0 immediately after $t = 0$ but it monotonically decreases and asymptotically approaches zero (i.e., $I_\infty = 0$) when $t \to \infty$ as the capacitor charge monotonically increases from $q_0 = 0$ toward its full charge q_∞.

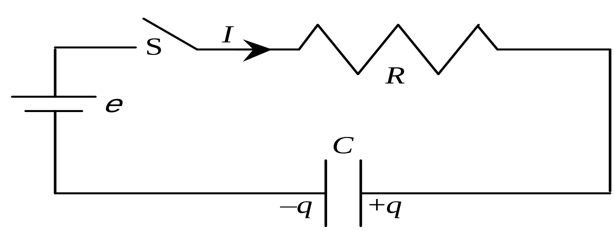

FIGURE 1. A series *RC* circuit connected to a battery ε through a switch S. The direction of I (after S is closed) has been chosen to be

consistent with the polarities of the terminals of the battery and capacitor.

To find expressions for I_0 and q_∞ in terms of the circuit parameters in Fig. 1, observe that the potential rise ε across the battery is always equal to the sum of the potential drops V_R and V_C respectively across the resistor and capacitor (after the switch has been closed) according to Kirchhoff's voltage loop rule. Initially $q_0 = 0$ and thus $V_{C0} = q_0 / C = 0$. Consequently $V_{R0} = \varepsilon$ so that $I_0 = V_{R0} / R = \varepsilon / R$. After a long time, $I_\infty = 0$ and hence $V_{R\infty} = I_\infty R = 0$. That implies $V_{C\infty} = \varepsilon$ and therefore $q_\infty = CV_{C\infty} = C\varepsilon$.

It remains to determine the time τ that characterizes how long it takes for the charge and current to approach these final values. In principle it takes an infinite amount of time for them to fully reach their final values [2] and so that is not a useful measure of τ (which is called the *time constant* of the circuit). By instead asking for the time it takes the charge to reach any definite fraction f (say 0.9) of q_∞ (or equivalently for the current to fall by 0.9 to 0.1 of I_0) then the answer will be finite. The exact number adopted for f can be freely chosen, as long as one uses it consistently.

The current is equal to the rate of change of charge on the capacitor plates. That is, $I = \Delta q / \Delta t \Rightarrow \Delta t = \Delta q / I$. This equation is only exact for an interval of time over which the current is constant. In the case of a charging *RC* circuit, the current is never constant but instead monotonically decreases to zero. However, if the current did remain constant at its initial value I_0 then the capacitor would fully charge up in a time equal to $(q_\infty - q_0) / I_0 = RC$ which therefore characterizes the charging process and can be *defined* to be τ. The time constant is proportional to C because a larger capacitance means a larger final charge q_∞ (for a given battery voltage ε) which will require more time to accumulate, but the rate of flow of charge is limited by the resistance and thus a larger R will mean it takes longer to achieve the final charge so that τ must also be proportional to R.

It is a useful student exercise to verify that the units properly work out as $\Omega \cdot \mathrm{F} = \mathrm{s}$. Dimensional reasoning thereby implies that no matter what value of f one adopts (between 0 and 1) it will be the case that $\tau \propto RC$. To show specifically that $f = 1 - 1/e \approx 0.63$ when $\tau = RC$, the exponential function must be introduced [3] but that can be deferred until a full mathematical analysis is presented.

B. Series *LR*

Next consider the *LR* circuit obtained by replacing the capacitor C in Fig. 1 with an inductor L. The inductor prevents any sudden changes in the current and so immediately after closing the switch S the current remains zero, $I_0 = 0$. Consequently $V_{R0} = I_0 R = 0$ across the resistor, which implies $V_{L0} = \varepsilon$ from the loop rule, but the flux in the inductor will initially remain $\Phi_0 = LI_0 = 0$. After a long time, the current will level off and stop changing, so that $V_{L\infty} = 0$ because there is no induced emf across an inductor when the flux in it is constant. In that case $V_{R\infty} = \varepsilon$ and thus $I_\infty = V_{R\infty} / R = \varepsilon / R$ and $\Phi_\infty = LI_\infty = \varepsilon L / R$.

However, the voltage across the inductor is $V_L = L\Delta I / \Delta t \Rightarrow \Delta t = L\Delta I / V_L$. This equation is only valid over a time interval during which V_L is constant, which is never true for an *LR* circuit. But by following the same kind of reasoning as for the *RC* circuit, if the inductor voltage remained constant at its initial value V_{L0} then the flux would reach its final value in a time of $L(I_\infty - I_0)/V_{L0} = L/R$ which can be defined to be the time constant τ. Here the inductive time constant is inversely proportional to R (in striking contrast to the capacitive time constant which is directly proportional to R) because a larger resistance implies a smaller final current I_∞ (for a given battery voltage) which will thus take less time to achieve, but the rate of change of current is limited by the inductance and thus a larger L will require a longer time to attain the final current so that $\tau \propto L$. Again, students should check that the units are such that $\mathrm{H} / \Omega = \mathrm{s}$.

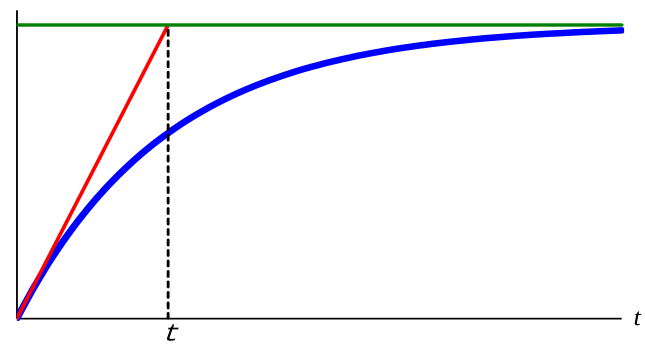

FIGURE 2. Curve (in blue) describing the rise and asymptotic leveling off of a quantity such as the capacitor charge q in an *RC* circuit after the switch S in Fig. 1 is closed at $t = 0$. Given that it becomes progressively harder to add more charge to a capacitor, the slope of this curve monotonically decreases to zero. A tangent line (in red) through the initial point (at the origin) intersects the final asymptote (in green) at a time of τ. If the vertical axis plots q, this statement is equivalent to the equation $I_0 = q_\infty / \tau$.

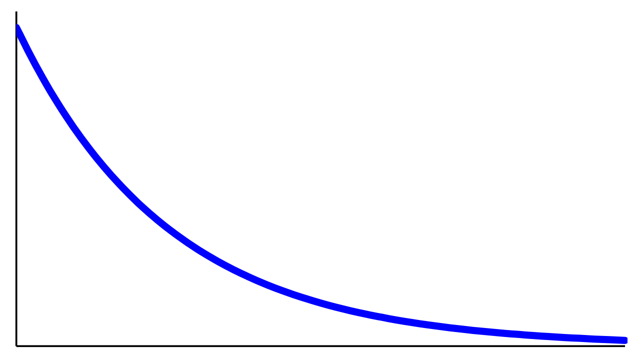

FIGURE 3. Curve describing the decrease and asymptotic leveling off to zero of a quantity such as the current I in an *RC* circuit after the switch S in Fig. 1 is closed at $t = 0$. Given that I is the time rate of change of q, this curve plots the point-by-point slope of the blue curve in Fig. 2.

III. CONCLUSIONS

In closing, students can conceptually be led to understand why q (or equivalently V_C) for a charging RC circuit must have a graph that looks like Fig. 2, whereas I (or equivalently V_R) must have a curve in the shape of Fig. 3. Likewise the rising flux and current (or equivalently the resistor voltage) of an LR circuit follows the plot in Fig. 2, but the rate of change of current (which is proportional to the inductor voltage) resembles Fig. 3. Students will now be primed to accept the subsequent mathematical facts that Fig. 3 is a decaying exponential of the form $\exp(-t/\tau)$ which equals 1 at $t = 0$ and whose argument is correctly dimensionless, whereas Fig. 2 must be the same curve flipped over (so that the two voltages add up to the constant battery emf) and is therefore unity minus that function which is a saturating exponential $1-\exp(-t/\tau)$.

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